The Grid Method of Discretisation for Type-2 Fuzzy Sets

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***Abstract*— In order to perform fuzzy inferencing, it is normal practice to discretise fuzzy sets. For type-1 fuzzy sets there is only one method of discretisation, but type-2 fuzzy sets may be discretised in more than one way. This paper introduces the grid method of type-2 discretisation — a simpler, more convenient alternative to the established technique.**

I. Introduction

Conventionally, discretisation is the first step in creating a computer representation of a fuzzy set (of any type). It is the process by which a continuous set is converted into a discrete set through a process of slicing. The rationale for discretisation is that a computer can process a finite number of slices, whilst it is unable to process the continuous fuzzy sets from which the slices are taken. For the ordinary type-1 fuzzy sets, there is only one dimension to be discretised, and hence only one method of discretisation. For type-2 fuzzy sets (T2FSs) a technique has become established which we term the *standard method* of type-2 discretisation. This paper presents the *grid method* of type-2 discretisation, which is an alternative to the standard method.

The paper is structured as follows: The remainder of the introduction describes fuzzy sets, concentrating on the type-2 fuzzy set. The next section concerns their application in fuzzy inferencing systems. Finally Section III focusses on the standard and grid methods of discretisation, the two main discretisation options for type-2 fuzzy sets.

A. Fuzzy Set Theory

Fuzzy set theory was originated in the 1960s by Lotfi Zadeh [9]. A fuzzy set is a set that does not have sharp boundaries, so allowing for *degree of truth*. Truth-values form a continuum on a scale from 0 to 1, with 0 representing *false*, and 1 representing *true*. Every fuzzy set is associated with a *membership function*; it is through its membership function that a fuzzy set is defined. The membership function maps each element of the domain onto its *degree of membership*, i.e. its truth-value. These are the 'ordinary' or *type-1* fuzzy sets, an example of which is depicted in Figure 1.

Definition 1 (Type-1 Fuzzy Set): Let X be a universe of discourse. A type-1 fuzzy set A on X is characterised by a membership function $\mu_A : X \to [0, 1]$ and can be formulated as follows [9]:

$$A = \{(x, \mu_A(x))|\ \mu_A(x) \in [0, 1]\ \forall x \in X\}. \qquad (1)$$

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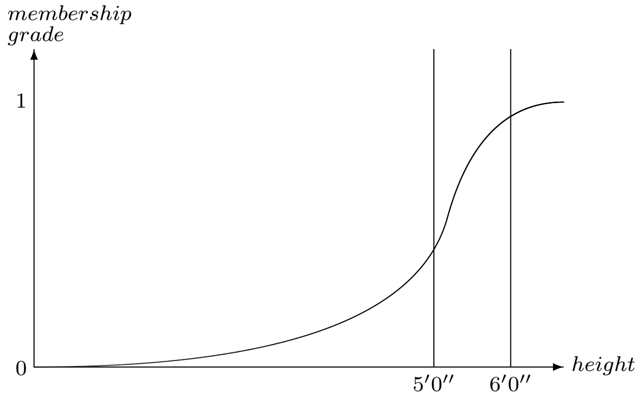

Fig. 1. Membership function for the type-1 fuzzy set *tall*.

1) Type-2 Fuzzy Sets: Type-2 fuzzy sets have elements whose membership grades are themselves fuzzy sets (of type-1). The graph of a type-2 fuzzy set is 3-dimensional; a typical such set is depicted in Figure 2. With no loss of generality it is assumed in the discussion which follows, that the type-2 fuzzy set is a surface represented by (x, u, z) co-ordinates contained within a unit cube (Figure 2[1]).

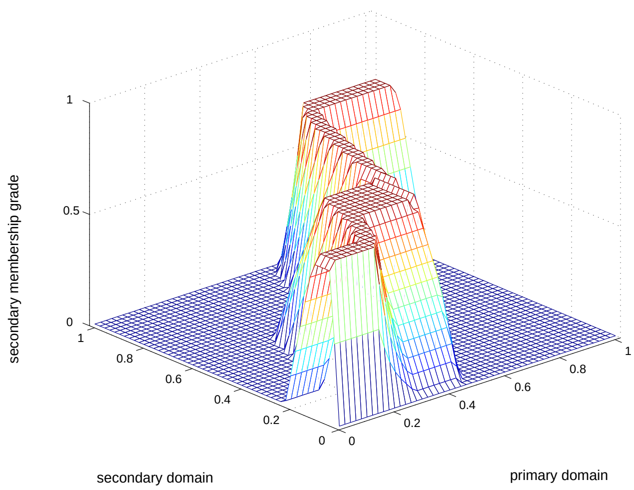

Fig. 2. A type-2 fuzzy set, produced by the aggregation stage of an FIS.

Let $U = [0, 1]$; let $\tilde{P}(U)$ be the set of fuzzy sets in U. A type-2 fuzzy set $\tilde{A}$ in X is a fuzzy set whose membership grades are themselves fuzzy [10], [11], [12]. This implies that $\mu_{\tilde{A}}(x)$ is a fuzzy set in U for all x, i.e. $\mu_{\tilde{A}} : X \to \tilde{P}(U)$

[1]The 'plateau' effect evident in this figure derives from the truncation of secondary membership functions during inferencing.

and

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | \; \mu_{\tilde{A}}(x) \in \tilde{P}(U) \; \forall x \in X\}. \qquad (2)$$

It follows that $\forall x \in X \; \exists J_x \subseteq U$ such that $\mu_{\tilde{A}}(x) : J_x \rightarrow U$. Applying (1), we have:

$$\mu_{\tilde{A}}(x) = \{(u, \mu_{\tilde{A}}(x)(u)) | \; \mu_{\tilde{A}}(x)(u) \in U \; \forall u \in J_x \subseteq U\}. \qquad (3)$$

X is called the primary domain and J_x the primary membership of x while U is known as the secondary domain and $\mu_{\tilde{A}}(x)$ the secondary membership of x.

Putting (2) and (3) together we obtain the mathematical definition of a type-2 fuzzy set:

Definition 2 (Type-2 Fuzzy Set):

$$\tilde{A} = \{(x, (u, \mu_{\tilde{A}}(x)(u))) | \; \mu_{\tilde{A}}(x)(u) \in U, \forall x \in X \wedge \forall u \in J_x \subseteq U\}. \qquad (4)$$

Some concepts associated with type-2 fuzzy sets are:

Definition 3 (Footprint Of Uncertainty): The *Footprint Of Uncertainty (FOU)* of a type-2 fuzzy set is the projection of the set onto the $x - u$ plane.

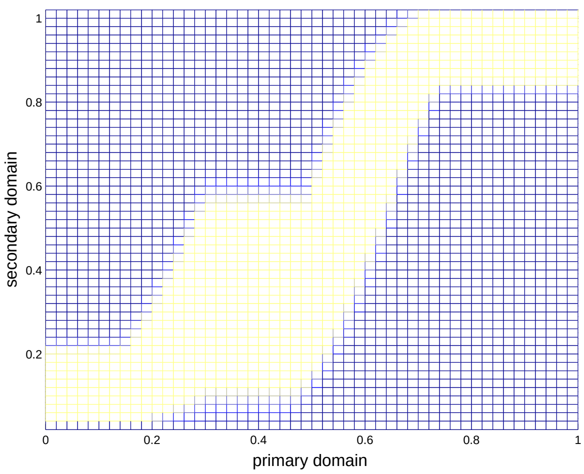

Fig. 3. The FOU of the type-2 fuzzy set depicted in Figure 2, i.e. the set in Figure 2 at a secondary membership level of 0.

Definition 4 (Lower Membership Function): The *Lower Membership Function (LMF)* of a type-2 fuzzy set is the type-1 membership function associated with the lower bound of the FOU.

Definition 5 (Upper Membership Function): The *Upper Membership Function (UMF)* of a type-2 fuzzy set is the type-1 membership function associated with the upper bound of the FOU.

Figure 4 depicts an FOU formed by blurring a type-1 fuzzy set, showing two vertical slices (Definition 7) at domain values x_1 and x_2. Figures 5 and 6 show triangular secondary membership functions at the vertical slices x_1 and x_2.

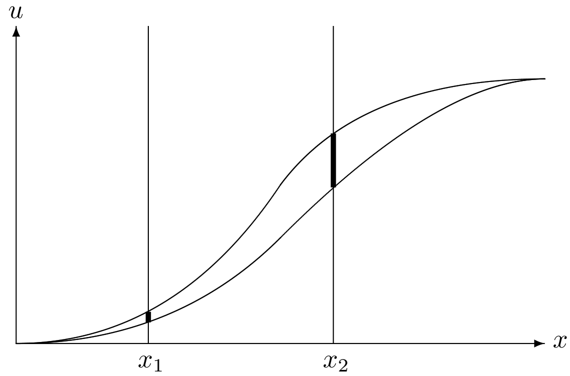

Fig. 4. Footprint of uncertainty with two vertical slices at x_1 and x_2. J_{x_1} and J_{x_2} are represented by the bold sections of the corresponding vertical slices.

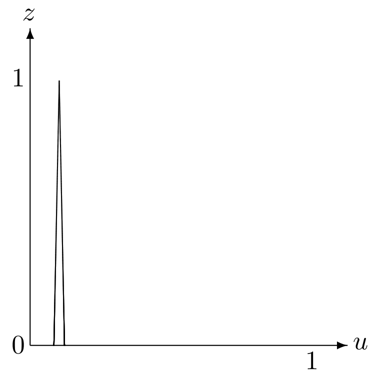

Fig. 5. Triangular secondary membership function at the vertical slice x_1 (Figure 4).

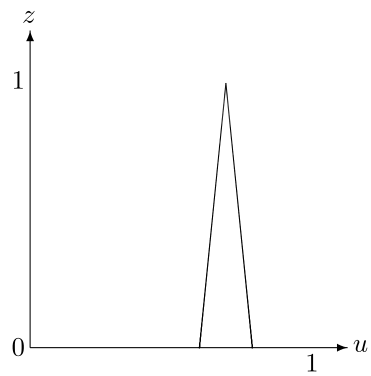

Fig. 6. Triangular secondary membership function of the vertical slice x_2 (Figure 4).

II. Fuzzy Inferencing Systems

The main employment of fuzzy sets is within a FIS. Starting with a crisp number, a Mamdani FIS[2] (of any type) passes through three stages: fuzzification, inferencing, and finally defuzzification:

Fuzzification is the process by which the degree of membership of a fuzzy set is determined, based on the crisp input value and the membership function of the fuzzy set.

Inferencing is the main stage of the FIS and may be broken down into three further stages: 1) antecedent computation, 2) implication, and 3) aggregation. The output of inferencing is a fuzzy set known as the *aggregated set*.

Defuzzification: During this stage the aggregated set is converted into another crisp number, the final result of the processing of the FIS.

a) Type-2 Mamdani FISs: Figure 7 provides a representation of a *type-2* Mamdani-style FIS, showing the defuzzification stage as consisting of two parts, *type-reduction* and defuzzification proper. Type-reduction is the procedure by which a type-2 fuzzy set is converted to a type-1 fuzzy set known as the *Type-Reduced Set (TRS)*. This set is then defuzzified to give a crisp number. The additional stage of type-reduction distinguishes the type-2 FIS from its type-1 counterpart.

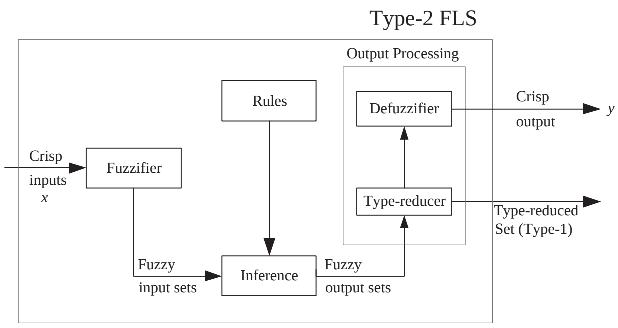

Fig. 7. Type-2 FIS (from Mendel [7]).

III. Discretisation of Type-2 Fuzzy Sets

Before discussing the topic of discretisation of type-2 fuzzy sets, we present some relevant definitions.

Definition 6 (Slice): A *slice* of a type-2 fuzzy set is a plane either

1) through the x-axis, parallel to the $u-z$ plane, or
2) through the u-axis, parallel to the $x-z$ plane.

Definition 7 (Vertical Slice [8]): A *vertical slice* of a type-2 fuzzy set is a plane through the x-axis, parallel to the $u-z$ plane.

Definition 8 (Degree of Discretisation): The *degree of discretisation* is the separation of the slices.

[2]This paper concentrates on the Mamdani type-2 FIS, though the observations regarding discretisation would apply equally to the alternative Takagi-Sugeno-Kang FIS.

The two strategies available for the discretisation of type-2 fuzzy sets are the standard method and the grid method.

A. Standard Method of Discretisation

In this discretisation technique (Figure 8) the primary domain of the type-2 fuzzy set is sliced vertically at even intervals. Each of the slices generated intersects the FOU; each line of intersection (within the FOU) is itself sliced at even intervals parallel to the $x-z$ plane. This results in different secondary domain degrees of discretisation according to the vertical slice [6]. The primary degree of discretisation and the number of horizontal slices are arbitrary, context dependent parameters, chosen by the developer after considering factors such as the power of the computer. Algorithm 1 shows the calculations involved in creating the computer representation of a T2FS by the standard method.

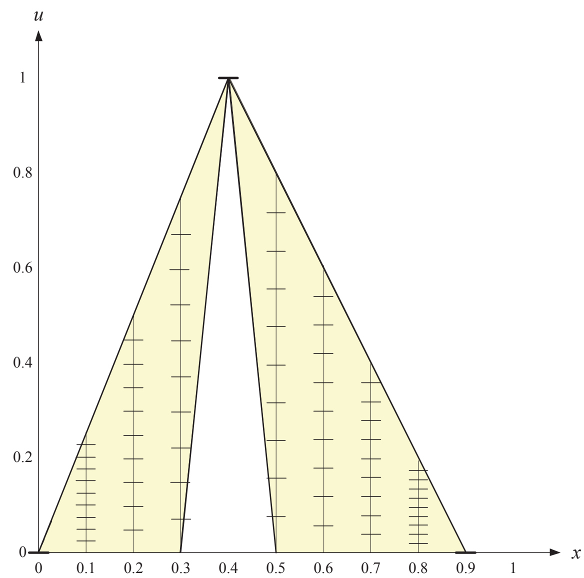

Fig. 8. $x-u$ plane under the standard method of discretisation. Primary domain degree of discretisation $= 0.1$.

Input: the parameters relating to a generalised T2FS
Output: a T2FS discretised using the standard method

calculate the primary domain of the T2FS;
vertically slice the primary domain at even intervals;
determine the number of horizontal slices required;
forall the *vertical slices* **do**
    calculate the secondary domain of the vertical slice;
    slice the secondary domain evenly;
    calculate the secondary grade for each intersection point (x, u);
end

Algorithm 1: Creation of a discretised T2FS by means of the standard method of discretisation.

B. Grid Method of Discretisation

An original, alternative method valid for all type-2 fuzzy sets is the *grid method of discretisation* (Figure 9). Since its first use in 2005 [5], this method has been employed on several more occasions [4], [2], [3]. In this approach the $x-u$ plane, $[0, 1]^2$, is evenly divided into a rectangular grid, as determined by the degrees of discretisation of the x and u-axes[3]. The fuzzy set surface, consisting of the secondary membership grades corresponding to each grid point (x, u) in the FOU, may be represented by a matrix of the secondary grades, in which the x and u co-ordinates are implied by the secondary grade's position within the matrix [1]. For grid points outside the FOU, a value of 0 is stored in the matrix. Figure 10 show a type-2 fuzzy set constructed with both LMF and UMF based on the S-curve. Figures 11 to 13 depict aggregated type-2 fuzzy sets created during the inferencing stage of FISs operating under the grid method of discretisation. Each fuzzy set is shown with its FOU and associated matrix. (Note that the orientation of the FOU is inverted relative to the matrix.)

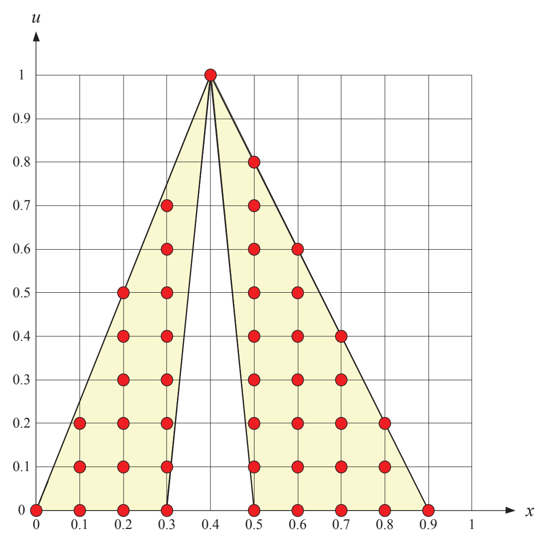

Fig. 9. $x-u$ plane under the grid method of discretisation. Primary domain and secondary domain degrees of discretisation are 0.1.

C. Critique of the Standard and Grid Methods

The grid method has certain advantages over the standard method:

1) Conceptually, the grid approach is very straightforward and easy to understand.
2) The grid approach confers a data structure on the type-2 fuzzy set. The set is represented by a rectangular matrix, which encapsulates the surface of the set (Figures 10(c), 11(c), 12(c) and 13(c)). Therefore the set does not need to be constructed from its membership functions as with the standard method.

[3]The degrees of discretisation of the axes are not necessarily the same.

3) The grid method involves a 2-D data structure, whereas the standard method deals with unstructured 3-D data.
4) The grid method is more general than the standard method in that it is applied to the whole theoretical domains X and U, i.e. it is not necessary to identify the actual domains of the membership function $\mu_{\tilde{A}}$ and the secondary membership functions $\mu_{\tilde{A}}(x)$ prior to discretisation.
5) Processing is simple using the grid method compared with the computationally complex standard method (Algorithm 1).
6) By employing the grid method, join and meet operations may be optimised [4].

However, a drawback of the grid method is that if the FOU has a narrow section, the discretisation has to be made finer in order to represent the type-2 fuzzy set adequately.

Summary

In this paper we have looked at the computer representation of T2FSs in the context of the FIS. The standard and grid methods of type-2 discretisation have been explored, and the grid method shown to have several advantages over the standard method, offset by only one disadvantage.

Further Work: Practical comparison of the standard and grid methods for accuracy and efficiency would be valuable.

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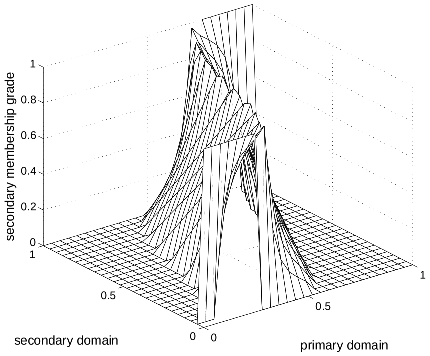
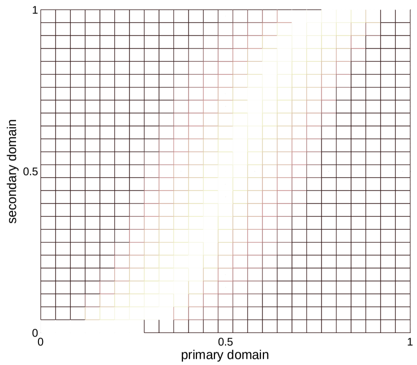


(a) 3-D representation.

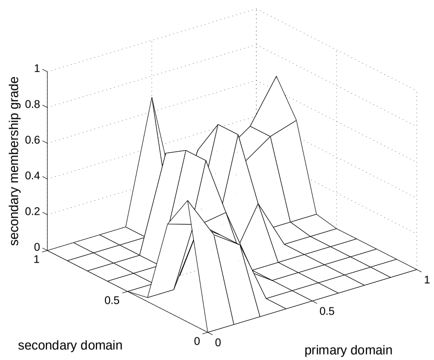


(b) FOU.

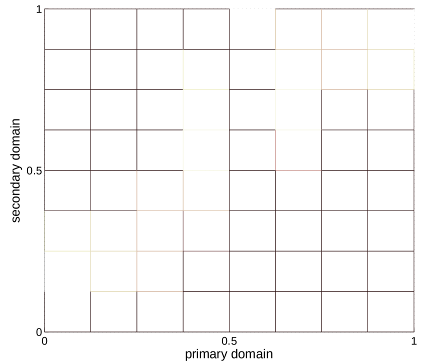
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.5868	0.7778	0.8615	0.9055	0.9424	0.9988	0.4132	0.2222	0.1385	0.0945	0.0576	0.0012	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.1736	0.5556	0.7230	0.8110	0.8722	0.9393	0.8264	0.4444	0.2770	0.1890	0.1278	0.0607	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.3333	0.5845	0.7164	0.8020	0.8798	0.9679	0.6667	0.4155	0.2836	0.1980	0.1202	0.0321	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.1111	0.4460	0.6219	0.7317	0.8202	0.9126	0.8889	0.5540	0.3781	0.2683	0.1798	0.0874	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.3075	0.5274	0.6615	0.7607	0.8573	0.9723	0.6925	0.4726	0.3385	0.2393	0.1427	0.0277	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.1690	0.4329	0.5913	0.7012	0.8020	0.9170	0.8310	0.5671	0.4087	0.2988	0.1980	0.0830	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0305	0.3384	0.5211	0.6417	0.7467	0.8617	0.9695	0.6616	0.4789	0.3583	0.2533	0.1383	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2439	0.4508	0.5821	0.6914	0.8064	0.9536	0.7561	0.5492	0.4179	0.3086	0.1936	0.0464	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.1493	0.3806	0.5226	0.6361	0.7511	0.8940	0.8507	0.6194	0.4774	0.3639	0.2489	0.1060	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0548	0.3104	0.4631	0.5808	0.6958	0.8345	0.9452	0.6896	0.5369	0.4192	0.3042	0.1655	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2402	0.4036	0.5254	0.6405	0.7750	0.9705	0.7598	0.5964	0.4746	0.3595	0.2250	0.0295	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.1699	0.3440	0.4701	0.5852	0.7155	0.9003	0.8301	0.6560	0.5299	0.4148	0.2845	0.0997	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0997	0.2845	0.4148	0.5299	0.6560	0.8301	0.9003	0.7155	0.5852	0.4701	0.3440	0.1699	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0295	0.2250	0.3595	0.4746	0.5964	0.7598	0.9705	0.7750	0.6405	0.5254	0.4036	0.2402	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.1655	0.3042	0.4192	0.5369	0.6896	0.9452	0.8345	0.6958	0.5808	0.4631	0.3104	0.0548	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.1060	0.2489	0.3639	0.4774	0.6194	0.8507	0.8940	0.7511	0.6361	0.5226	0.3806	0.1493	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0464	0.1936	0.3086	0.4179	0.5492	0.7561	0.9536	0.8064	0.6914	0.5821	0.4508	0.2439	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.1383	0.2533	0.3583	0.4789	0.6616	0.9695	0.8617	0.7467	0.6417	0.5211	0.3384	0.0305	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0830	0.1980	0.2988	0.4087	0.5671	0.8310	0.9170	0.8020	0.7012	0.5913	0.4329	0.1690	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0277	0.1427	0.2393	0.3385	0.4726	0.6925	0.9723	0.8573	0.7607	0.6615	0.5274	0.3075	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0874	0.1798	0.2683	0.3781	0.5540	0.8889	0.9126	0.8202	0.7317	0.6219	0.4460	0.1111	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0321	0.1202	0.1980	0.2836	0.4155	0.6667	0.9679	0.8798	0.8020	0.7164	0.5845	0.3333	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0607	0.1278	0.1890	0.2770	0.4444	0.8264	0.9393	0.8722	0.8110	0.7230	0.5556	0.1736	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0012	0.0576	0.0945	0.1385	0.2222	0.4132	0.9988	0.9424	0.9055	0.8615	0.7778	0.5868	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

(c) Matrix.

Fig. 10. Type-2 fuzzy set with S-curve LMF and UMF, triangular secondary membership functions, discretised by the grid method with domain and codomain degrees of discretisation 0.04.



(a) 3-D representation.

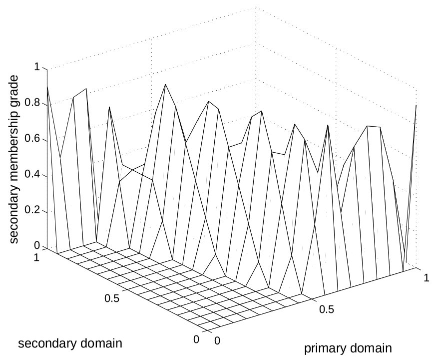


(b) FOU.

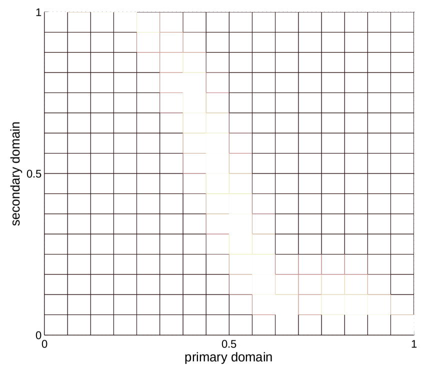
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.6806	0.4450	0.3438	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.4913	0.4450	0.3438	0.1003	0.0000	0.0000	0.0000	0.0000	0.0000
0.0231	0.0231	0.3438	0.3673	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.6000	0.0000	0.2714	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.6000	0.0000	0.6000	0.0231	0.0231	0.0231
0.0000	0.0000	0.0000	0.5268	0.0000	0.6000	0.3438	0.4450	0.4913
0.0000	0.0000	0.0000	0.0000	0.0000	0.4000	0.3438	0.4450	0.6806
0.0000	0.0000	0.0000	0.0000	0.6806	0.0000	0.0000	0.0000	0.0000

(c) Matrix.

Fig. 11. FIS generated aggregated set. The grid method of discretisation, with domain and codomain degrees of discretisation 0.125, was used throughout the FIS.



(a) 3-D representation.

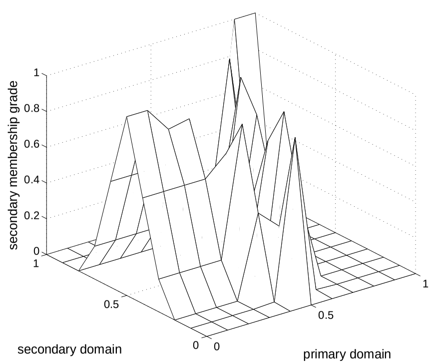


(b) FOU.

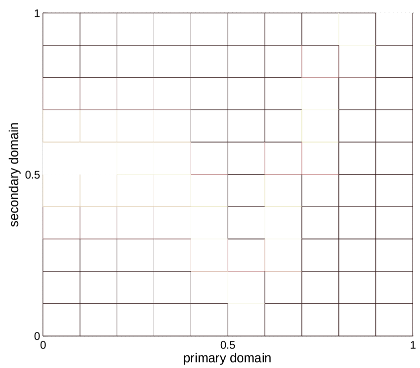
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.9096
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.3661	0.9024	0.3875	0.7344	0.8296	0.8014	0.4847	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.1875	0.8125	0.6054	0.3875	0.6054	0.6054	0.4450	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.4479	0.8711	0.3011	0.3011	0.3011	0.3011	0.0886	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.1148	0.7083	0.6708	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.3000	0.9096	0.4625	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.4852	0.8472	0.2542	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0929	0.6704	0.6736	0.0458	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2714	0.8556	0.5000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.4500	0.8711	0.3264	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.6286	0.7458	0.1528	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.1875	0.8071	0.5375	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.3875	0.9024	0.3292	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.3875	0.7125	0.1208	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.3438	0.3875	0.4000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.7344	0.3875	0.0875	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.9096	0.4847	0.8014	0.8296	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

(c) Matrix.

Fig. 12. FIS generated aggregated set. The grid method of discretisation, with domain and codomain degrees of discretisation 0.0625, was used throughout the FIS.



(a) 3-D representation.



(b) FOU.

0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0056	0.8916	0.0056	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.4576	0.3495	0.4576	0.0000	0.0000	0.0000	0.0000
0.1833	0.1833	0.1833	0.1833	0.9096	0.0000	0.9096	0.0000	0.0000	0.0000	0.0000
0.5917	0.5917	0.5917	0.5917	0.7001	0.0000	0.7001	0.0173	0.0000	0.0000	0.0000
1.0000	1.0000	0.8594	0.8832	0.3253	0.0000	0.3253	0.3683	0.0000	0.0000	0.0000
0.5917	0.5917	0.5917	0.5917	0.0000	0.0000	0.0000	0.7192	0.0000	0.0000	0.0000
0.1833	0.1833	0.1833	0.1833	0.0000	0.0000	0.0000	0.8832	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2992	0.1563	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.8594	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	1.0000

(c) Matrix.

Fig. 13. FIS generated aggregated set. The grid method of discretisation, with domain and codomain degrees of discretisation 0.1, was used throughout the FIS.