Defuzzification of the Discretised Generalised Type-2 Fuzzy Set: Experimental Evaluation☆

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Abstract

The work reported in this paper addresses the challenge of the efficient and accurate defuzzification of discretised generalised type-2 fuzzy sets as created by the inference stage of a Mamdani Fuzzy Inferencing System. The exhaustive method of defuzzification for type-2 fuzzy sets is extremely slow, owing to its enormous computational complexity. Several approximate methods have been devised in response to this defuzzification bottleneck. In this paper we begin by surveying the main alternative strategies for defuzzifying a *generalised* type-2 fuzzy set: (1) Vertical Slice Centroid Type-Reduction; (2) the sampling method; (3) the elite sampling method; and (4) the α-planes method. We then evaluate the different methods experimentally for accuracy and efficiency. For accuracy the exhaustive method is used as the standard. The test results are analysed statistically by means of the Wilcoxon Nonparametric Test and the elite sampling method shown to be the most accurate. In regards to efficiency, Vertical Slice Centroid Type-Reduction is demonstrated to be the fastest technique.

Keywords: Type-2 Fuzzy Set, Defuzzification, Sampling Method, α-Planes Method, VSCTR

1. Introduction

In this paper responses to the challenge of the efficient and accurate defuzzification of discretised generalised type-2 fuzzy sets are evaluated. Defuzzification is the crucial final stage of the five-stage Fuzzy Inferencing System (FIS) as illustrated in Figure 1. Type-2 defuzzification consists of two parts — *type-reduction* and defuzzification proper. Type-reduction is the procedure by which a type-2 fuzzy set is converted to a type-1 fuzzy set. This set is then defuzzified to give a crisp number. Owing to its enormous computational complexity, the additional stage of type-reduction of a *type-2* FIS has come to be regarded as a bottleneck [24]. The progress of generalised type-2 applications has been impeded as developers have opted [3, pages 7, 8, 16] for the computationally simpler interval type-2 FISs [38, 39] for which an increasing number of applications are being developed in areas such as control, simulation and optimisation [1, 2, 4–6, 20–22, 27, 29, 31, 32, 36, 41, 42, 52]. In contrast, there are relatively few, though varied, generalised type-2 fuzzy applications [24, 33, 39, 45].

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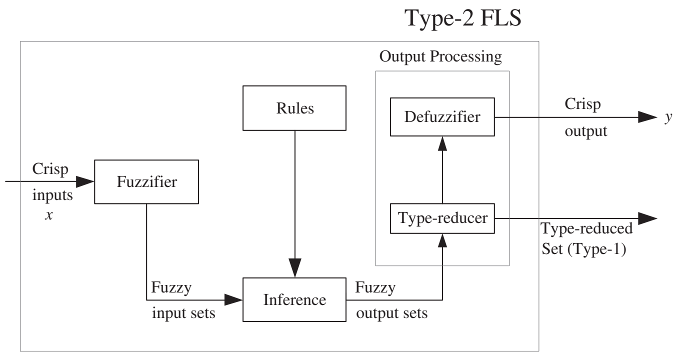


Figure 1: Type-2 FIS (from Mendel [37]).

The main strength of type-2 fuzzy logic is its ability to deal with the second-order uncertainties that arise from several sources [26], among them the fact that the meanings of words are often vague [39, page 117]. The Karnik-Mendel Iterative Procedure (KMIP) [25, 48] is the established technique for defuzzification of interval sets. The capability of the generalised type-2 paradigm to handle uncertainty is explored in [14]. Regrettably interval type-2 fuzzy sets are not able to model uncertainty as fully as their generalised counterparts, as they lack the crucial variability of the third dimension [39]. Our research, therefore, sees developing *generalised* type-2 systems as a challenge for the research community. A triangular type-2 system with a defuzzification algorithm based on the KMIP has been developed by Starczewski [46]; this goes some way towards achieving our goal. Coupland and John [8, 9] have exploited geometry to improve the speed of inferencing in generalised type-2 fuzzy sets. In 2008 Liu [34, 40] proposed the α-planes method which involves decomposing a generalised type-2 set into a set of α-planes, which are horizontal slices akin to interval type-2 sets. This method is used in conjunction with an interval method such as the KMIP, the Greenfield-Chiclana Collapsing Defuzzifier [15], or the Nie-Tan Method [44]. The α-planes/KMIP method has been modified by Zhai and Mendel [53] to increase its efficiency. Experiments have shown that the α-planes decomposition introduces slight inaccuracies [18]; this is touched on in Section 7. Further inaccuracies are introduced by the associated interval method, as all the alternatives (apart from the interval exhaustive method) are approximations [15]. Independently, Wagner and Hagras have introduced the notion of zSlices [47], a concept similar to α-planes.

Table 1 shows the development of the field of type-2 defuzzification over the past decade, as reflected in the major publications. A number of researchers have been working simultaneously and independently in this field, and the solutions developed are diverse and original. The application developer now has a choice of several methods; the stage has been reached where an experimental evaluation of the methods is desirable so as to establish the best performing method in the generalised case. Such an evaluation is the motivation behind this paper.

In this paper we shall be focussing on the predominant *discretised* type-2 FIS as created by the inference stage of a Mamdani FIS.[1]. The exhaustive method of defuzzification for type-2 fuzzy sets is extremely slow, owing to its enormous computational complexity. Several approximate methods have been devised in response to this defuzzification bottleneck. We begin by surveying the main alternative strategies for defuzzifying a *generalised* type-2 fuzzy

[1]Most FISs rely on discretisation, though a non-discretised FIS has been realised: Coupland and John [8, 9] have exploited geometry to improve the speed of inferencing in generalised type-2 fuzzy sets.

DATE	AUTHORS	METHOD	REFERENCE	PUBLISHER/PUBLICATION
2001	Jerry M. Mendel	**Exhaustive**	[37]	Prentice-Hall PTR
February 2001	Nilesh N. Karnik Jerry M. Mendel	KMIP	[25]	Information Sciences
October 2002	Hongwei Wu Jerry M. Mendel	Wu-Mendel Approximation	[50]	IEEE Transactions on Fuzzy Systems
July 2007	Luís Alberto Lucas Tania M. Centeno Myriam R. Delgado	**VSCTR**	[35]	Proc. FUZZ-IEEE 2007
June 2008	Maowen Nie Woei Wan Tan	Nie-Tan	[44]	Proc. FUZZ-IEEE 2008
June 2008	Sarah Greenfield Robert I. John	Stratified TRS	[13]	Proc. IPMU 2008
May 2008	Feilong Liu	**α-Planes Representation**	[34]	Information Sciences
June 2009	Sarah Greenfield Francisco Chiclana Simon Coupland Robert I. John	Collapsing	[15]	Information Sciences
July 2009	Sarah Greenfield Francisco Chiclana Robert I. John	CORL	[17]	Proc. IFSA-EUSFLAT 2009
June 2011	Dongrui Wu Maowen Nie	EIASC	[49]	Proc. FUZZ-IEEE 2011
July 2011	Francisco Chiclana Shang-Ming Zhou	Type-1 OWA	[7]	Proc. EUSFLAT-LFA 2011
April 2012	Sarah Greenfield Francisco Chiclana Robert I. John Simon Coupland	**Sampling**	[19]	Information Sciences

Table 1: Chronology of publication of defuzzification methods. The methods shown in bold are included in the evaluation reported below.

set: (1) Vertical Slice Centroid Type-Reduction; (2) the sampling method; (3) the elite sampling method; and (4) the α-planes method. It is timely that these techniques are evaluated experimentally; in this paper we report on how we have done this in relation to accuracy and efficiency. For accuracy the exhaustive method is used as the standard. The test results for accuracy are analysed statistically by means of the Wilcoxon Nonparametric Test. In regards to efficiency, timings are compared to establish the fastest technique.

For the research reported in this paper it is assumed (1) the type-2 fuzzy set is contained within a unit cube, (2) the type-2 fuzzy set may be viewed as a surface represented by (x,u,z) co-ordinates[2], and (3) for type-1 sets the centroid method of defuzzification [28, page 336] is employed.

1.1. Mathematical Definition of the Type-2 Fuzzy Set

Let X be a universe of discourse. A fuzzy set A in X is characterised by a membership function $\mu_A : X \to [0,1]$. A fuzzy set A in X can be expressed as follows:

$$A = \{(x, \mu_A(x)); \mu_A(x) \in [0,1] \, \forall x \in X\}. \tag{1}$$

Note that the membership grades of A are crisp numbers.

Let $\tilde{P}(X)$ be the set of fuzzy sets in X. A type-2 fuzzy set $\tilde{A}$ in X is a fuzzy set whose membership grades are themselves fuzzy. This implies that $\mu_{\tilde{A}}(x)$ is a fuzzy set in $[0,1]$ for all x, i.e.

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)); \mu_{\tilde{A}}(x) \in \tilde{P}([0,1]) \, \forall x \in X\}. \tag{2}$$

This implies that $\forall x \in X \, \exists J_x \subseteq [0,1]$ such that $\mu_{\tilde{A}}(x) : J_x \to [0,1]$. Applying (1), we have:

$$\mu_A(x) = \{(u, \mu_{\tilde{A}}(x)(u)); \mu_{\tilde{A}}(x)(u) \in [0,1] \, \forall u \in J_x \subseteq [0,1]\}. \tag{3}$$

J_x is called the primary membership of x while $\mu_{\tilde{A}}(x)$ is called the secondary membership of x.

Putting (2) and (3) together we have

$$\tilde{A} = \{(x, (u, \mu_{\tilde{A}}(x)(u))) | \mu_{\tilde{A}}(x)(u) \in [0,1], \, \forall x \in X \wedge \forall u \in J_x \subseteq [0,1]\}. \tag{4}$$

This 'vertical representation' of a type-2 fuzzy set is used to define the concept of an *embedded set* of a type-2 fuzzy set, which is fundamental to the definition of the *centroid* of a type-2 fuzzy set.

2. Discretisation

Conventionally, discretisation is the first step in creating a computer representation of a fuzzy set (of any type). It is the process by which a continuous set is converted into a discrete set through a process of slicing. The rationale for discretisation is that a computer can process a finite number of slices, whilst it is unable to process the continuous fuzzy sets from which the slices are taken.

Definition 1 (Slice)**.** *A* slice *of a type-2 fuzzy set is a plane either*

1. *through the x-axis, parallel to the $u-z$ plane, or*

2. *through the u-axis, parallel to the $x-z$ plane.*

[2] This paper is concerned solely with fuzzy sets for which the (primary) domain is numeric in nature.

Definition 2 (Vertical Slice [39])**.** *A* vertical slice *of a type-2 fuzzy set is a plane through the x-axis, parallel to the $u-z$ plane.*

Definition 3 (Degree of Discretisation)**.** *The* degree of discretisation *is the separation of the slices.*

For a type-2 fuzzy set, both the primary and secondary domains are discretised, the former into vertical slices. The primary and secondary domains, which are both the unit interval $U = [0,1]$, may have different degrees of discretisation. Furthermore the secondary domain's degree of discretisation is not necessarily constant. For type-2 fuzzy sets there is more than one discretisation strategy [11]. In the experimental evaluation reported below, we employ the grid method of discretisation [11].

3. Defuzzification of Generalised Type-2 Fuzzy Sets

For type-1 fuzzy sets defuzzification is a straightforward matter. There are several defuzzification techniques available, including the centroid, centre of maxima and mean of maxima [30]. Type-2 defuzzification of discretised type-2 fuzzy sets is a process that consists of two stages [37]:

1. Type-reduction, which converts a type-2 fuzzy set to a type-1 fuzzy set, and
2. defuzzification of the type-1 fuzzy set.

Mathematically, the type-reduction algorithm depends upon the *Extension Principle* [51], which generalises operations defined for crisp numbers to type-1 fuzzy sets. Type-2 defuzzification techniques therefore derive from and incorporate type-1 defuzzification methods[3].

3.1. The Wavy-Slice Representation Theorem

The concept of an *embedded type-2 fuzzy set (embedded set)* or *wavy-slice* [39] is crucial to type-reduction. An embedded set is a special kind of type-2 fuzzy set. It relates to the type-2 fuzzy set in which it is embedded in this way: For every primary domain value, x, there is a unique secondary domain value, u, plus the associated secondary membership grade that is determined by the primary and secondary domain values, $\mu_{\tilde{A}}(x)(u)$.

Example 1. *In Figure 2 we have identified two embedded sets of a type-2 fuzzy set with primary and secondary domain degree of discretisation of* 0.1*. The embedded set $\tilde{P}$ is represented by pentagonal, pointed flags, and embedded set $\tilde{Q}$ is symbolised by quadrilateral shaped flags.*

We can represent these embedded sets as sets of points, thus:

$$\tilde{P} = \{[0.1/0]/0 + [0.1/0.1]/0.1 + [0.5/0.4]/0.2 + [0.5/0.1]/0.3 + [1/1]/0.4 + [0.9/0.6]/0.5 + [0.4/0]/0.6 + [0.4/0.2]/0.7 + [0.2/0.2]/0.8 + [0.1/0]/0.9\}.$$

$$\tilde{Q} = \{[0.1/0]/0 + [0.2/0]/0.1 + [0.5/0.1]/0.2 + [0.5/0.6]/0.3 + [1/1]/0.4 + [0.8/0.7]/0.5 + [0.5/0.3]/0.6 + [0.5/0.1]/0.7 + [0.3/0.1]/0.8 + [0.1/0]/0.9\}.$$

[3] Geometric defuzzification [9] is exceptional among type-2 defuzzification methods in not involving type-reduction and therefore not requiring type-1 defuzzification.

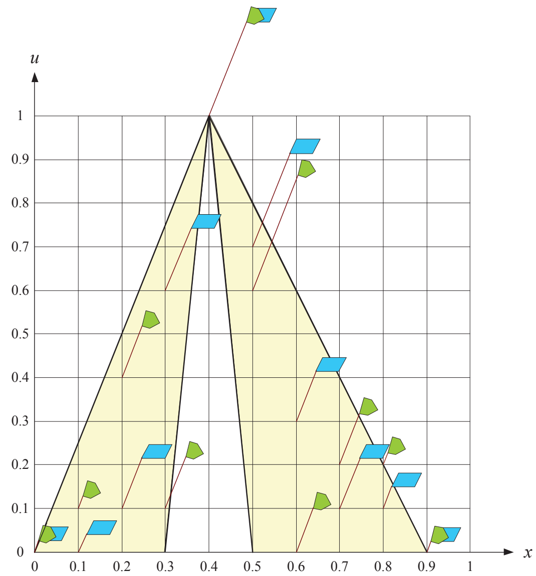


Figure 2: Two embedded sets, indicated by different flag styles. The flag height reflects the secondary membership grade. Degree of discretisation of primary and secondary domains is 0.1. The shaded region is the FOU.

Definition 4 (Embedded Set)**.** *Let $\tilde{A}$ be a type-2 fuzzy set in X. For discrete universes of discourse X and U,* an embedded type-2 set *$\tilde{A}_e$ of $\tilde{A}$ is defined as the following type-2 fuzzy set*

$$\tilde{A}_e = \{(x_i, (u_i, \mu_{\tilde{A}}(x_i)(u_i)))| \, \forall i \in \{1, \dots, N\} : \, x_i \in X \;\; u_i \in J_{x_i} \subseteq U\}. \tag{5}$$

$\tilde{A}_e$ contains exactly one element from J_{x_1}, J_{x_2}, ..., J_{x_N}, namely u_1, u_2, ..., u_N, each with its associated secondary grade, namely $\mu_{\tilde{A}}(x_1)(u_1)$, $\mu_{\tilde{A}}(x_2)(u_2)$, ..., $\mu_{\tilde{A}}(x_N)(u_N)$.

Mendel and John have shown that a type-2 fuzzy set can be represented as the union of its type-2 embedded sets [39, page 121]. This powerful result is known as the type-2 fuzzy set *Representation Theorem* or *Wavy-Slice Representation Theorem*; in [39] it was derived without reference to the Extension Principle. Bringing a conceptual simplicity to the manipulation of type-2 fuzzy sets, it is applied to give simpler derivations of results previously obtained through the Extension Principle [39].

The Representation Theorem is formally stated thus [39, page 121]:

> Let $\tilde{A}_e^j$ denote the jth type-2 embedded set for type-2 fuzzy set $\tilde{A}$, i.e.,
>
> $$\tilde{A}_e^j \equiv \left\{ \left(u_i^j, \mu_{\tilde{A}}(x_i)(u_i^j) \right), i = 1, \dots, N \right\}$$
>
> where $\{u_i^j, \dots, u_N^j\} \in J_{x_i}$. Then $\tilde{A}$ can be represented as the union of its type-2 embedded sets, i.e.,
>
> $$\tilde{A} = \sum_{j=1}^{n} \tilde{A}_e^j$$
>
> where
>
> $$n \equiv \prod_{i=1}^{N} M_i.$$

We regard the exhaustive defuzzification algorithm as the standard by which other algorithms must be evaluated.

The first stage of type-2 defuzzification is to create the Type-Reduced Set (TRS). Assuming that the primary domain X has been discretised, the TRS of a type-2 fuzzy set may be defined through the application of Zadeh's Extension Principle [51]. Alternatively the TRS may be defined via the Representation Theorem [39, page 121].

Definition 5. *The TRS associated with a type-2 fuzzy set $\tilde{A}$ with primary domain X discretised into N points is*

$$C_{\tilde{A}} = \left\{ \left(\frac{\sum_{i=1}^{N} x_i \cdot u_i}{\sum_{i=1}^{N} u_i}, \mu_{\tilde{A}}(x_1)(u_1) * \ldots * \mu_{\tilde{A}}(x_N)(u_N) \right) \middle| \; \forall i \in \{1, \ldots, N\} : \; x_i \in X \;\; u_i \in J_{x_i} \subseteq U \right\}. \tag{6}$$

The type reduction stage requires the application of a t-norm ($*$) to the secondary membership grades. Because the product t-norm does not produce meaningful results for type-2 fuzzy sets with general secondary membership functions[4] it is to be avoided. For the work presented in this paper, the minimum t-norm is used.

In order for this definition of the TRS to be meaningful, the domain X must be numeric in nature. The TRS is a type-1 fuzzy set in U and its computation in practice requires the secondary domain U to be discretised as well. Algorithm 1 (adapted from Mendel [37]) is used to compute the TRS of a type-2 fuzzy set.

3.2. Exhaustive Type-Reduction

Mendel and John's Representation Theorem (Subsection 3.1) provides a precise, straightforward method for type-2 defuzzification. Though Definition 5 does not explicitly mention embedded sets, they appear implicitly in Equation 6. When this equation is presented in algorithmic form (Algorithm 1), explicit mention is made of embedded sets. As *every* embedded set is processed, this strategy has become known as the *exhaustive method* [16]. Discretisation inevitably brings with it an element of approximation. However the exhaustive method does not introduce further inaccuracies subsequent to discretisation.

Exhaustive type-reduction processes every embedded set in turn. Each embedded set is defuzzified as a type-1 fuzzy set. The defuzzified value is paired with the minimum secondary membership grade of the embedded set. The set of ordered pairs constitutes the TRS.

4. Efficient Generalised Type-Reduction Strategies

4.1. The Sampling Method

In response to the computational bottleneck engendered by exhaustive defuzzification, the sampling method, also known as the *sampling defuzzifier* [19], was devised as a cut-down version of the exhaustive method. Instead of all the embedded sets participating in type-reduction, a sample is randomly selected in order to derive an approximation for the defuzzified value. Associated with continuous type-2 fuzzy sets are an infinite number of embedded sets, and therefore the centroid values obtained via Algorithm 1 are in fact estimates of the real centroid values. Consequently discretisation in itself may be seen as a form of sampling of the continuous type-2 fuzzy set.

[4] Under the product t-norm, $\lim_{N\to\infty} [\mu_{\tilde{A}}(x_1)(u_1) * \ldots * \mu_{\tilde{A}}(x_N)(u_N)] = 0$ [25, page 201].

Input: a discretised generalised type-2 fuzzy set
Output: a discrete type-1 fuzzy set (the TRS)

forall the *embedded sets* **do**
  find the minimum secondary membership grade (z) ;
  calculate the primary domain value (x) of the type-1 centroid of the type-2 embedded set ;
  pair the secondary grade (z) with the primary domain value (x) to give set of ordered pairs (x,z) $\{$some values of x may correspond to more than one value of $z\}$;
end
forall the *primary domain (x) values* **do**
  select the maximum secondary grade $\{$make each x correspond to a unique secondary domain value$\}$;
end

Algorithm 1: Type-reduction of a discretised type-2 fuzzy set to a type-1 fuzzy set, adapted from Mendel [37].

Random Selection of an Embedded Set. Because the enumeration of all the possible embedded sets is not practical, a process of *random construction* is employed to sample them. For each primary domain value, a certain number of secondary domain (u) values lie within the FOU. For the grid method of discretisation, these are located at the grid intersections within the FOU. The construction of an embedded set requires the selection of a secondary domain (u) value for each primary domain value. For each primary domain value, secondary domain values are selected using a random function, and therefore have the same probability of being chosen. This selection method ensures that the subsets of n embedded sets as described above constitute a random sample, but the embedded sets are not guaranteed to be unique.

User Selected Parameters. The **sample size**, i.e. the number of embedded sets, is a parameter selected by the user. A higher number of embedded sets will result in a better accuracy of defuzzification results. The **primary and secondary degrees of discretisation** are also user selected parameters. They are normally pre-selected prior to the invocation of the FIS.

The Sampling Algorithm. The user having selected the necessary parameters, the embedded sets are randomly selected and processed (Algorithm 2). The sampling method, despite having the extra stages indicated in the algorithm, is radically simpler computationally than the exhaustive method.

4.2. The Elite Sampling Method

The sampling algorithm (Algorithm 2) allows a given domain value to be associated with more than one secondary grade. However in *elite sampling* (Algorithm 3), each domain value is associated with only one membership grade, that being the maximum secondary grade available to the domain value (as with exhaustive type-reduction). Elite sampling, though more computationally complex than basic sampling, is designed to be more accurate in situations where there are a significant number of redundant embedded sets in the sample.

Input: a discretised generalised type-2 fuzzy set
Output: a discrete type-1 fuzzy set

select the primary domain degree of discretisation {normally pre-selected} ;
select the secondary domain degree of discretisation {normally pre-selected} ;
select the sample size ;
repeat
    randomly select (i.e. construct) an embedded set ;
    process the embedded set according to steps 2 to 4 of Algorithm 1 ;
until *the sample size is reached*;

Algorithm 2: TRS obtained through sampling (in conjunction with the grid method of discretisation).

Input: a discretised generalised type-2 fuzzy set
Output: a discrete type-1 fuzzy set

select the primary domain degree of discretisation {normally pre-selected} ;
select the secondary domain degree of discretisation {normally pre-selected} ;
select the sample size ;
repeat
    randomly select (i.e. construct) an embedded set ;
    process the embedded set according to steps 2 to 4 of Algorithm 1 ;
until *the sample size is reached*;
forall the *primary domain (x) values* **do**
    select the maximum secondary grade {make each x correspond to a unique secondary domain value} ;
end

Algorithm 3: TRS obtained through elite sampling (in conjunction with the grid method of discretisation).

4.3. Vertical Slice Centroid Type-Reduction

Vertical Slice Centroid Type-Reduction (VSCTR) is a highly intuitive[5] method employed by John [23]; the paper of Lucas et al. [35] renewed interest in this strategy. In this approach the type-2 fuzzy set is cut into vertical slices, each of which is defuzzified as a type-1 fuzzy set (Algorithm 4). By pairing the domain value with the defuzzified value of the vertical slice, a type-1 fuzzy set is formed, which is easily defuzzified to give the defuzzified value of the type-2 fuzzy set. Though chronologically preceding it, this method is a generalisation of the Nie-Tan method for interval type-2 fuzzy sets [44].

Input: a discretised generalised type-2 fuzzy set
Output: a discrete type-1 fuzzy set (the TRS)

forall the *vertical slices* **do**
  find the defuzzified value using the centroid method ;
  pair the domain value of the vertical slice with the defuzzified value to give set of ordered pairs (i.e. a type-1 fuzzy set) ;
end

Algorithm 4: VSCTR of a discretised type-2 fuzzy set to a type-1 fuzzy set.

4.4. The α-Plane Representation

In 2008 Liu [34, 40] proposed the α-planes representation. By this technique a generalised type-2 fuzzy set is decomposed into a set of α-planes, which are horizontal slices akin to interval type-2 fuzzy sets. By repeated application of an interval defuzzification method, Liu [34] has shown that a generalised type-2 fuzzy set may be type-reduced. This method of type-reduction (Algorithm 5) is depicted in Figure 3. By defuzzifying the resultant type-1 fuzzy set, the defuzzified value for the generalised type-2 fuzzy set is obtained.

Though the α-plane representation was envisaged as being used with the Karnik-Mendel Iterative Procedure (KMIP) [34], any interval method may be used. Any variation on the KMIP, such as the Enhanced Iterative Algorithm with Stop Condition [49] will locate the endpoints of the TRS interval. Other interval methods, such as the Greenfield-Chiclana Collapsing Defuzzifier [15], or the Nie-Tan Method [44], will defuzzify the α-plane; their defuzzified values (which will be located approximately in the centre of the interval) may then be formed into the type-1 TRS. In [16] the most accurate interval method was shown to be *collapsing outward right-left (CORL)*. CORL is therefore the interval technique chosen to be associated with the α-planes method for the experimental evaluation reported in Section 5.

Independently to Liu, and at about the same time, Wagner and Hagras introduced the notion of zSlices [47], a concept very similar to α-planes. The α-planes/KMIP method has been modified by Zhai and Mendel [53] to increase its efficiency.

5. Experimental Comparison

5.1. Test Sets

Six FIS generated generalised type-2 fuzzy test sets were created[6], depicted in Figures 4 to 9. These are aggregated sets produced by the inferencing stage of *Fuzzer*, a prototype type-2

[5]No mathematical justification has been provided to show that VSCTR leads to the same defuzzified value as the exhaustive method.

[6]The initial intention was to include Liu's two generalised type-2 fuzzy test sets [34, pages 2230 – 2233]. However this was not feasible, since (1) for Case A the secondary membership functions are derived by a random

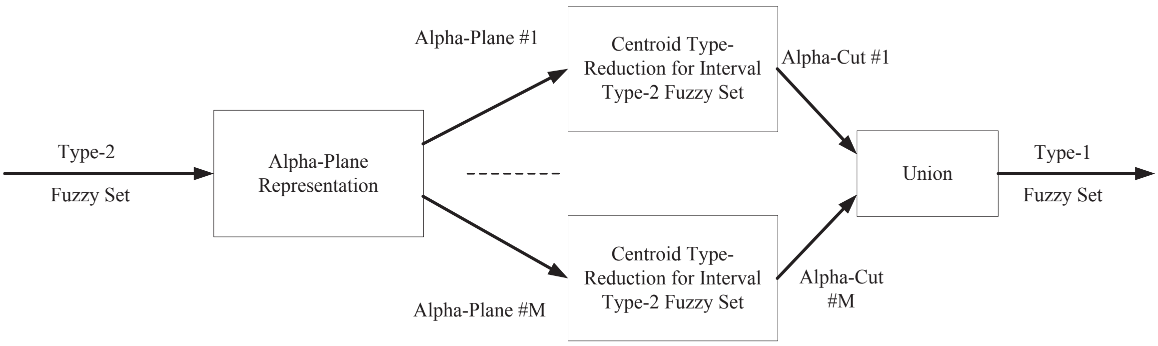


Figure 3: Defuzzification using the α-Planes Representation (from Liu [34]).

Input: a discretised generalised type-2 fuzzy set
Output: a discrete type-1 fuzzy set

decompose the type-2 fuzzy set into α-planes ;
forall the *α-planes* **do**
    find the left and right endpoints using the KMIP ;
    pair each endpoint with the α-plane height to give set of ordered pairs (i.e. a type-1 fuzzy set) {each α-plane is paired with two endpoints } ;
end

Algorithm 5: Type-reduction of a type-2 fuzzy set to a type-1 fuzzy set using the α-plane method.

FIS [10]. For each inference the degree of discretisation adopted was sufficiently coarse to allow exhaustive defuzzification; without the benchmark defuzzified values obtained through exhaustive defuzzification, the methods could not have been compared for accuracy. Three rule sets were used. For each rule set the FIS was run with two distinct sets of parameters[7]. The FIS generated test sets were chosen because of the complexity and lack of symmetry evident in their graphs; their benchmark defuzzified values were found by exhaustive defuzzification. The three rule sets are shown in Tables 2 to 4. Table 5 contains a summary of the features of the test sets.

Heater FIS This FIS is designed to calculate the desirable setting for a heater. It has 5 rules and 2 inputs which are tabulated in Table 2.

Washing Powder FIS The purpose of this FIS is to determine the amount of washing powder required by a washing machine for a given wash load. It has 4 rules and 3 inputs which are summarized in Table 3.

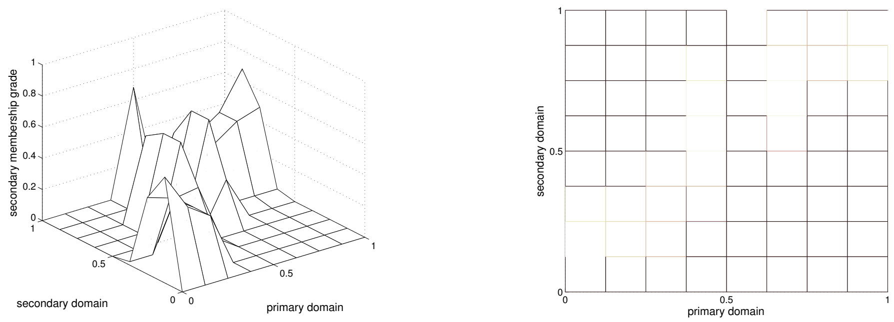
procedure and therefore cannot be recreated, and (2) in Case B the secondary membership functions are too similar to interval membership functions for this set to be of value as a generalised test set.

[7]For example Heater0p0625 is not a finer version of Heater0p125; it uses different parameters for the input rules. That these two test sets are completely different can be clearly seen from their 3D representations (Appendices A to F).

Shopping FIS This FIS is designed to answer the dilemma of whether to go shopping by car, or walk, depending on weather conditions, amount of shopping, etc.. The defuzzified value is therefore rounded to one of two possible answers. The FIS has 4 rules and 3 inputs as tabulated in Table 4.

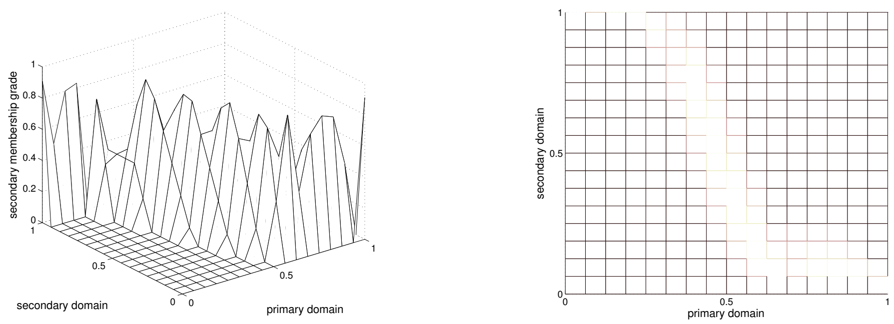
INPUTS		**OUTPUTS**
TEMPERATURE	DATE	HEATING
cold	—	high
—	winter	high
hot	not winter	low
—	spring	medium
—	autumn	medium

Table 2: Heater FIS rules.



(a) 3-D representation. (b) FOU.

Figure 4: HeaterFIS0.125 — Heater FIS generated generalised test set, domain degree of discretisation 0.125.



(a) 3-D representation. (b) FOU.

Figure 5: HeaterFIS0.0625 — Heater FIS generated generalised test set, domain degree of discretisation 0.0625.

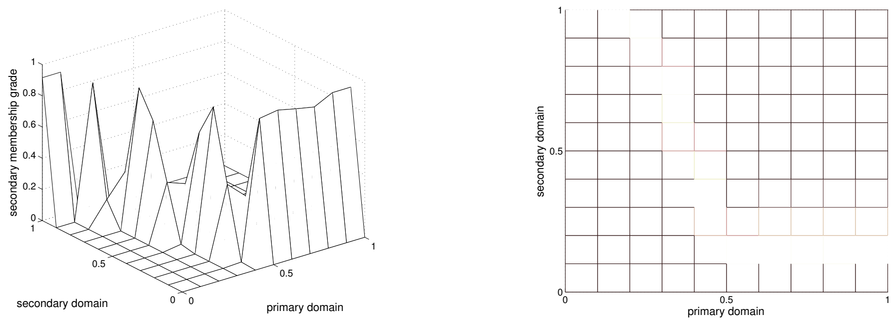
5.2. Methodology for Generalised Methods Comparison

The six test sets were defuzzified using the following techniques:

1. The exhaustive method (as a benchmark for accuracy),
2. VSCTR,

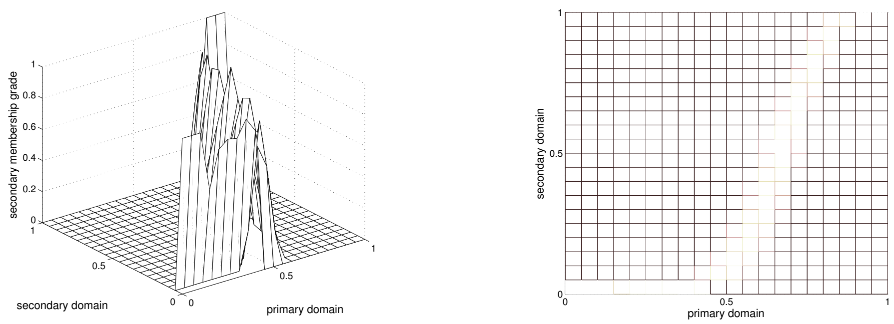
INPUTS			OUTPUTS
WASHING	WATER	PRE-SOAK	POWDER
very dirty	—	—	a lot
—	hard	—	a lot
slightly dirty	soft	—	a bit
—	—	lengthy	a bit

Table 3: Washing Powder FIS rules.



(a) 3-D representation. (b) FOU.

Figure 6: PowderFIS0.1 — Powder FIS generated generalised test set, domain degree of discretisation 0.1.

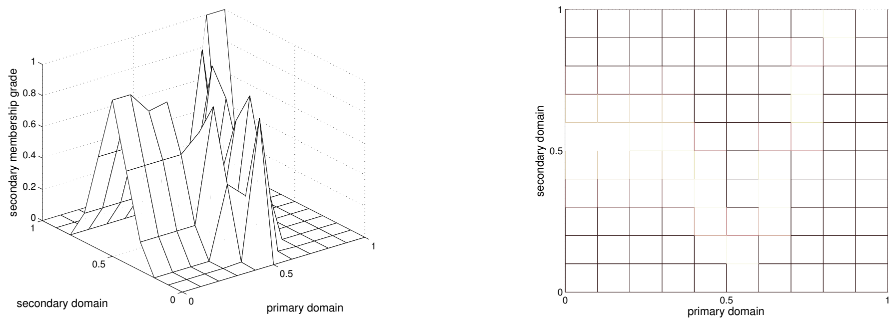


(a) 3-D representation. (b) FOU.

Figure 7: PowderFIS0.05 — Powder FIS generated generalised test set, domain degree of discretisation 0.05.

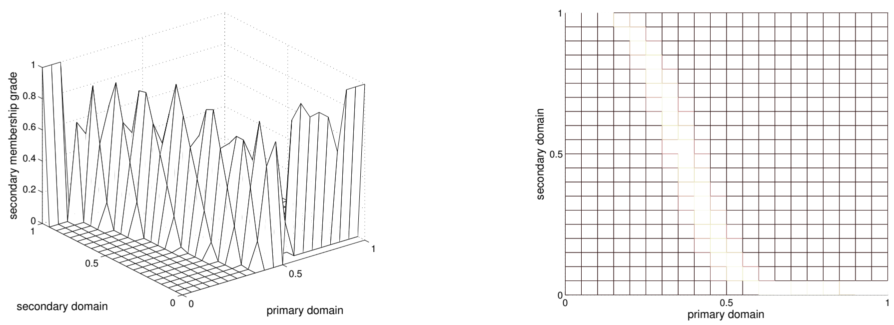
INPUTS			OUTPUTS
DISTANCE	SHOPPING	WEATHER	TRAVEL METHOD
short	light	—	walk
long	—	—	go by car
—	heavy	—	go by car
—	—	raining	go by car

Table 4: Shopping FIS rules.



(a) 3-D representation. (b) FOU.

Figure 8: ShoppingFIS0.1 — Shopping FIS generated generalised test set, domain degree of discretisation 0.1.



(a) 3-D representation. (b) FOU.

Figure 9: ShoppingFIS0.05 — Shopping FIS generated generalised test set, domain degree of discretisation 0.05.

TEST SET	NORMAL FOU	NORMAL SEC. MF	NARROW FOU	NO. OF EMB. SETS
HeaterFIS0.125	yes	no	no	14580
HeaterFIS0.0625	yes	no	yes	13778100
PowderFIS0.1	yes	no	yes	24300
PowderFIS0.05	yes	yes	yes	3840000
ShoppingFIS0.1	yes	yes	no	312500
ShoppingFIS0.05	yes	yes	yes	3840000

Table 5: Features of the generalised test sets.

3. the sampling method using sample sizes of 50, 100, 250, 500, 750, 1000, 5000, 10000, 50000 and 100000,
4. the elite sampling method using sample sizes of 50, 100, 250, 500, 750, 1000, 5000, 10000, 50000 and 100000,
5. the α-planes/CORL method using 3, 5, 9, 11, 21, 51, 101, 1001, 10001 and 100001 α-planes, and
6. the α-planes/Interval Exhaustive method using 3, 5, 9, 11, 21, 51, 101, 1001, 10001 and 100001 α-planes (as an evaluation of the accuracy of the α-planes representation itself)[8].

For each test run the defuzzified value and the defuzzification time were recorded. For the timings, in most instances, multiple runs were performed and the times averaged to give results of greater accuracy than those that would have been obtained from a single run[9].

The defuzzification methods were coded in Matlab^{TM} and tested on a laptop with an AMD Turion II Neo K645 CPU, a clock speed of 1.6 GHz, and a 4096MB 1333MHz Dual Channel DDR3 SDRAM, running the MS Windows®7 SP1 Home Premium 64 bit operating system. For timings, the defuzzification software was run as a process with priority higher than that of the operating system, so as to eliminate, as far as possible, timing errors caused by other operating system processes.

6. Discussion of the Test Results

The test results are tabulated in Appendices A to F (Tables A.6 to F.29), which record the defuzzified values, errors, and timings. The elite sampling tables (Tables A.8, B.12, C.16, D.20, E.24 and F.28) show data relating to non-redundant embedded sets[10]. The timings indicate that VSCTR is the most efficient method of those tested.

6.1. Statistical Comparison of the Methods

We now present a rigourous statistical analysis of the test results for accuracy.

The hypothesis that we are testing in this subsection can be stated as follows:

> *The sampling, elite sampling, VSCTR and α-planes/CORL methods do not produce significantly different defuzzified values.*

To compare each pair of methods we have to analyse two related samples, the defuzzified values obtained by each method's application to the same six test sets referred to above. The usual parametric test to use in these cases is the t-test applied to the difference scores. This test requires for its application the assumption of normality and independent distribution of the difference scores in the population from which the six test sets are drawn[11]. However, on the one hand, we consider these assumptions to be unjustifiable in our context since there is no evidence to support them, i.e. we have no information about the nature of the population from which the six test sets are drawn nor do we have any knowledge about any of its parameters. Also, by not requiring these stringent assumptions we can, on the other hand, achieve greater

[8]The extremely long processing times prevented defuzzification using 10001 and 100001 α-planes with test sets HeaterFIS0.0625, PowderFIS0.05 and ShoppingFIS0.05.

[9]In the minority of cases having a lengthy defuzzification time, only one timing was taken.

[10]A Non-Redundant Embedded Set (NRES) is an embedded set that is not eliminated during elite sampling.

[11]Although we did not apply any specific random sampling method, we consider the set of six test sets to constitute a sample representative of the whole set of generalised type-2 fuzzy sets.

generality in our conclusions. Therefore, we conclude that nonparametric tests are most appropriate in our experimental study; we will use the Wilcoxon Matched-Pairs Signed-Ranks Test [43] to be described in the next subsection.

6.1.1. Wilcoxon Matched-Pairs Signed-Ranks Statistical Test

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from some unknown continuous distribution function F. Let p be a positive real number, $0 < p < 1$, and let $\xi_p(F)$ denote the quantile of order p for the distribution function F, that is, $\xi_p(F)$ is a solution of $F(x) = p$. For $p = 0.5$, $\xi_{0.5}(F)$ is known as the median of F.

A problem of location is set up by testing the null hypothesis $H_0 : \xi_p(F) = \xi_0$ against one of the alternatives $\xi_p(F) > \xi_0$, $\xi_p(F) < \xi_0$ or $\xi_p(F) \neq \xi_0$. The Wilcoxon Signed-Ranks Test provides a statistical hypothesis test which takes into account the magnitude of the difference between the observations and the hypothesized quantile in order to solve the issue of location.

Let $H_0 : \xi_{0.5}(F) = \xi_0$ be the null hypothesis. Consider the differences $D_i = X_i - \xi_0$, $i = 1, 2, \ldots, n$. Under H_0, the expected number of negative differences will be $n/2$ and negative and positive differences of equal absolute magnitude should occur with equal probability. Consider the absolute values $|D_1|, |D_2|, \ldots, |D_n|$ and rank them from 1 to n. Let T_+ be the sum of ranks assigned to those $D_i's$ that are positive and T_- be the sum of ranks assigned to those $D_i's$ that are negative. It follows that

$$T_+ \; + T_- = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

so T_+ and T_- are linearly related and offer equivalent criteria. A large value of T_+ indicates that most of the larger ranks are assigned to positive $D_i's$. It follows that large values of T_+ support $H_1 : \xi_{0.5}(F) > \xi_0$. A similar analysis applies to the other two alternatives. So, the test rejects $H_0 : \xi_{0.5}(F) = \xi_0$ to accept $H_1 : \xi_{0.5}(F) > \xi_0$ if $T_+ > c_1$, it rejects H_0 to accept $H_1 : \xi_{0.5}(F) < \xi_0$ if $T_- > c_2$ and it rejects H_0 to accept $H_1 : \xi_{0.5}(F) \neq \xi_0$ if $T_+ > c_3$ or $T_- > c_4$ where c_i are the critical region values.

Under H_0, the common distribution of T_+ and T_- is symmetric with mean $E[T_+] = n(n+1)/4$ and variance $\mathrm{var}[T_+] = n(n+1)(2n+1)/24$. For large n, the standardized T_+ has approximately a standard normal distribution.

In the case of matched-paired data $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ obtained from the application of two treatments (in our case – two generalised defuzzification methods) to the same set of subjects (in our case – the set of six test sets), in order to test $H_0 : \xi_{0.5}(F_{X_i - Y_i}) = \xi_0$ against one-sided or two-sided alternatives, the Wilcoxon Test is performed exactly as above by taking $D_i = X_i - Y_i - \xi_0$. In our study we want to test whether the application of the different generalised defuzzification methods produces significantly different defuzzified values, i.e. we are testing a null hypothesis with a value $\xi_0 = 0$, $H_0 : \xi_{0.5}(F_{X_i - Y_i}) = 0$. We are testing against the alternative hypothesis of method X being more accurate than method Y, so we will use one-tailed testing $H_1 : \xi_{0.5}(F_{X_i - Y_i}) < 0$.

We assume that two measures with test p-value under the null hypothesis lower than or equal to 0.05 (α) will be considered as significantly different; we refer to it as the test being significant and therefore we conclude that the null hypothesis tested is to be rejected. Otherwise, we will fail to reject the null hypothesis.

6.1.2. Results of the Wilcoxon Signed Rank Test

We wanted to test whether there is a significant difference in accuracy between the four generalised methods. The methods may be paired in six ways. For the α-planes/CORL method we analysed the results obtained by using the highest number of α-planes (100001), so that

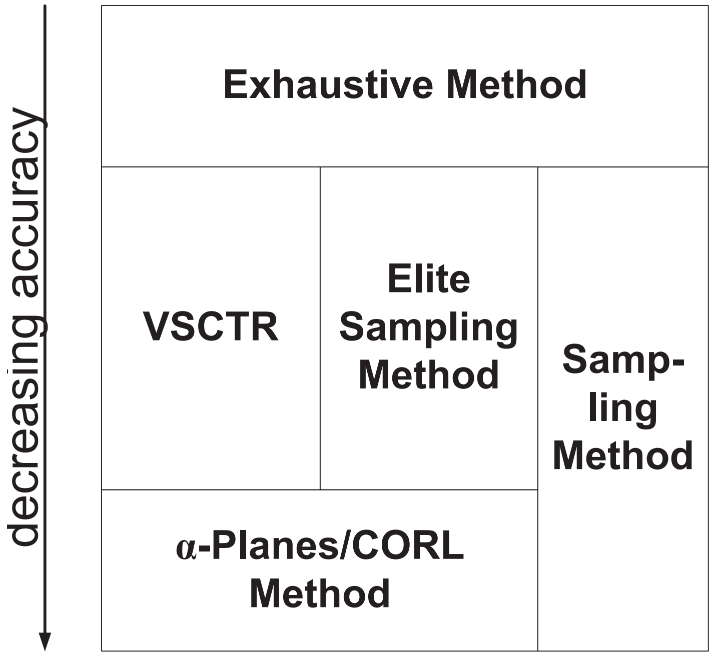


Figure 10: Hierarchy of type-2 defuzzification methods' performance in relation to accuracy, for sample sizes 50 and 100. The exhaustive method is used as a benchmark.

discretisation effects on the *u*-axis would be eliminated as far as possible. For the sampling and elite sampling methods, the Wilcoxon Tests were applied at the sample sizes used in the test runs.

The Wilcoxon Signed Rank Test results, presented in Tables G.30 to G.39, reveal a more complex and interesting picture than that revealed by simply ranking the test results. Taking each pair of comparisons in turn,

1. For every sample size the sampling and VSCTR methods do not produce significantly different defuzzified values.
2. For sample sizes up to and including 5000 the elite sampling and VSCTR methods do not produce significantly different defuzzified values. For sample sizes of 10000 and above there is evidence to support the elite sampling method being more accurate than VSCTR.
3. For sample sizes up to and including 1000 the sampling and elite sampling methods do not produce significantly different defuzzified values. For sample sizes over 5000 there is evidence to support the elite sampling method being more accurate than sampling method.
4. For every sample size there is evidence to support VSCTR being more accurate than the α-planes/CORL method.
5. For sample sizes of 50 and 100 the sampling and α-planes/CORL methods do not produce significantly different defuzzified values. For sample sizes of 250 and above there is evidence to support the sampling method being more accurate than the α-planes/CORL method.
6. For every sample size there is evidence to support the elite sampling method being more accurate than the α-planes/CORL method.

Figures 10 to 13 display these relative accuracies graphically.

7. Conclusions

Several conclusions may be drawn from this investigation:

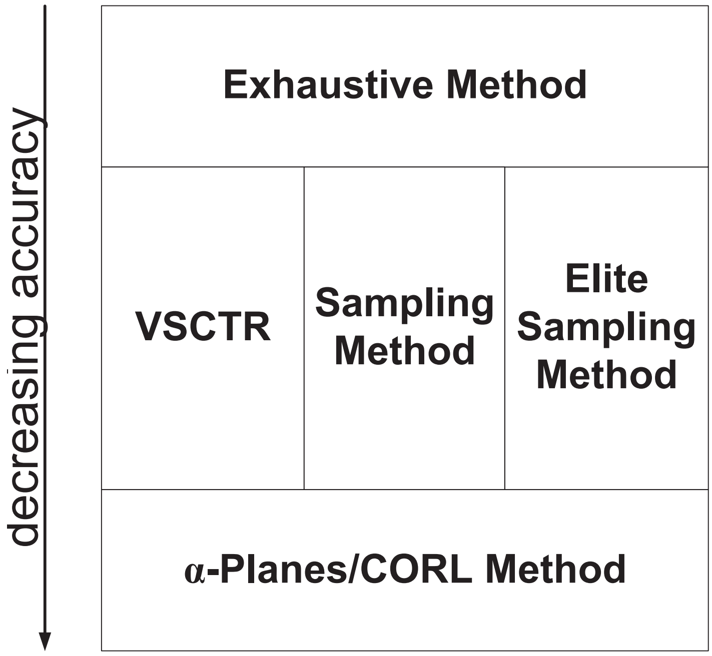


Figure 11: Hierarchy of type-2 defuzzification methods' performance in relation to accuracy, for sample sizes 250, 500, 750 and 1000. The exhaustive method is used as a benchmark.

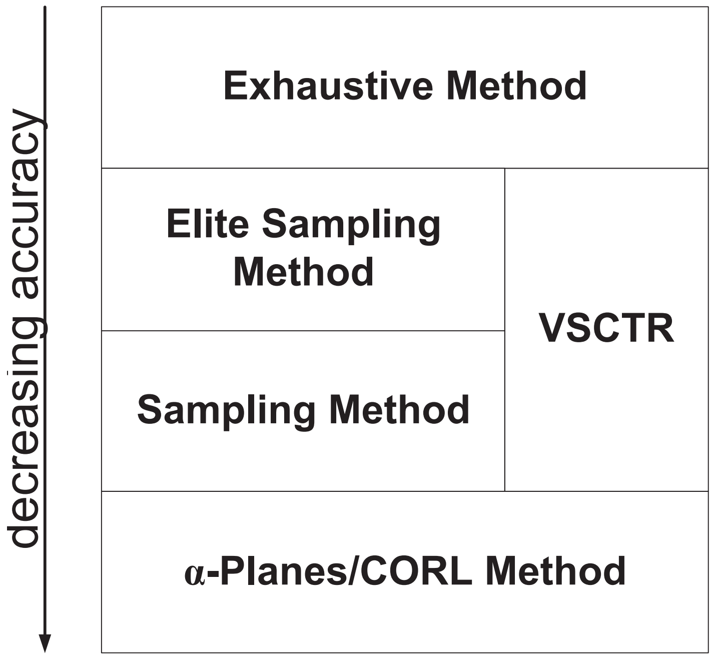


Figure 12: Hierarchy of type-2 defuzzification methods' performance in relation to accuracy, for sample size 5000. The exhaustive method is used as a benchmark.

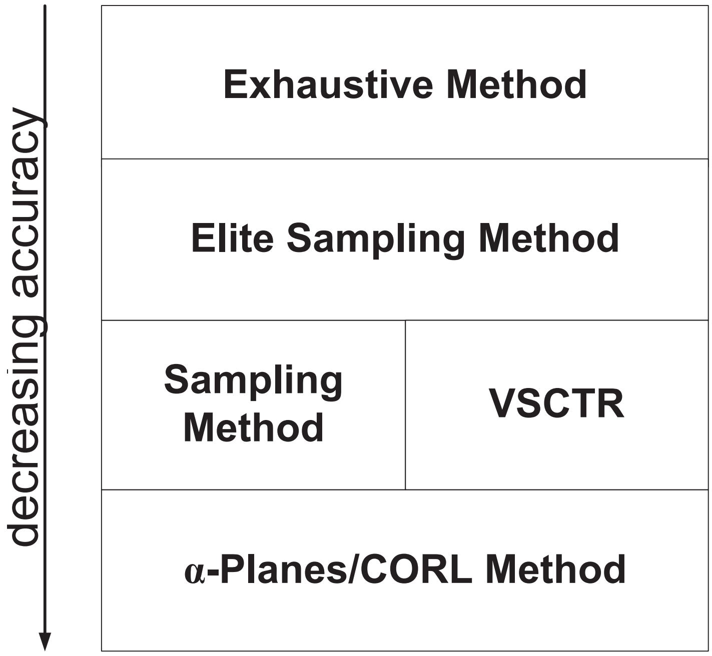


Figure 13: Hierarchy of type-2 defuzzification methods' performance in relation to accuracy, for sample sizes 10000 to 100000. The exhaustive method is used as a benchmark.

Most accurate method The experimental evaluation reveals a complex picture as regards the relative accuracies of the methods:

- The α-planes/CORL method is the least accurate of the techniques assessed apart from when compared with the sampling method at low sample sizes (50 and 100). In these instances, there is no evidence to support the sampling method being more accurate than the α-planes/CORL method.
- For sample sizes of 5000 and above elite sampling is more accurate than sampling[12].
- For high sample sizes (10000, 50000 and 100000) the elite sampling method is the most accurate of the techniques compared. However, with such high sample sizes, it may be argued that elite sampling is barely distinguishable from the exhaustive method, especially when the set to be defuzzified has a low number of embedded sets.
- For samples of moderate size (250, 500, 750 and 1000), VSCTR, sampling, and elite sampling are of equivalent accuracy.

Fastest method VSCTR is undoubtedly the fastest method for defuzzification of type-2 fuzzy sets; none of the other methods challenge VSCTR for speed, no matter how low the sample size in the case of the sampling method, or the number of α-planes employed by the α-planes method.

Convergence of the α-planes/CORL results As the number of α-planes increases, the α-planes/CORL results do not converge to the value obtained by generalised exhaustive defuzzification. Furthermore even the α-planes/interval exhaustive results (Tables A.9,

[12]Tables A.8, B.12, C.16, D.20, E.24 and F.28 show that there are numerous redundant embedded sets, which when eliminated from the calculation during elite sampling, leave few non-redundant embedded sets, making the effective sample size much smaller. For low sample sizes, elite sampling is not an improvement on sampling, but for higher sample sizes, elite sampling outperforms sampling, as even after the redundant embedded sets have been discarded, there are still sufficient to give a good approximation to the exhaustive defuzzified value.

C.17 and E.25) fail to converge to this value. The defuzzified values for both the α-planes/CORL and α-planes/interval exhaustive methods are similar (to a precision of about four decimal places) and appear to converge to the same number, which is *not* the value obtained from generalised exhaustive defuzzification. This discrepancy is indicative of an issue with the α-planes method itself, and has been previously reported in [18] and [12].

In summary, the results reported in this paper will motivate the development of generalised type-2 fuzzy applications. They will also help researchers in selecting the most most appropriate defuzzification method for the application of generalised type-2 fuzzy logic in areas such as perceptual computing [24], fuzzy logic control [39], diagnostic medicine [45] and clustering [33], among others.

8. Further Work

Out of the research presented in this paper, certain issues have emerged that would benefit from further work:

Standard Method of Discretisation Investigate the accuracy and efficiency of the generalised type-2 defuzzification methods when implemented using the standard method of discretisation. We would expect there to be far fewer, if any, NRESs, and that consequently elite sampling would outperform sampling for accuracy, even at low sample sizes.

α-Planes Method Investigate why the defuzzified value obtained through the α-planes method does not converge to the exhaustive defuzzified value as the number of α-planes is increased.

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Appendix A. Generalised Test Set Heater0.125

EXHAUSTIVE DEFUZZIFIED VALUE	NO. OF EMB. SETS	NO. OF NRESS	EXHAUST-IVE TIMING	VSCTR DEFUZZIFIED VALUE	VSCTR ERROR	VSCTR TIMING
0.6313618377	14580	486	1.37 secs.	0.6327431582	0.0013813205	0.000268 secs.

Table A.6: Exhaustive and VSCTR results for the HeaterFIS0.125 test set.

SAMPLE SIZE	PERCENT. OF EMB. SETS SAMPLED	SAMPLING DEFUZZIFIED VALUE	SAMPLING METHOD ERROR	SAMPLING METHOD TIMING
50	0.34%	0.6235041770	0.0078576607 ◁	0.0188 secs.
100	0.69%	0.6255373882	0.0058244495 ◁	0.0377 secs.
250	1.71%	0.6299440373	0.0014178004 ◁	0.0937 secs.
500	3.43%	0.6262109521	0.0051508856	0.188 secs.
750	5.14%	0.6263047645	0.0050570732	0.282 secs.
1000	6.86%	0.6246724480	0.0066893897	0.377 secs.
5000	34.29%	0.6251282506	0.0062335871	1.93 secs.
10000	68.59%	0.6256899730	0.0056718647	4.03 secs.
50000	342.94%	0.6252882201	0.0060736176	39.1 secs.
100000	685.87%	0.6254891164	0.0058727213	2.34 mins.

Table A.7: Sampling results for the HeaterFIS0.125 test set. Number of embedded sets = 14580. Percentage of embedded sets sampled = $\frac{\text{sample size}}{\text{number of embedded sets}} \times 100$. Exhaustive defuzzified value = 0.6313618377. Errors marked '◁' are lower than the corresponding errors for the elite sampling method.

SAMPLE SIZE	%-AGE OF ESS SAMPLED	NO. OF NRESS IN SAMPLE	NRESS AS %-AGE OF SAMP. SIZE	NRESS AS %-AGE OF ESS	ELITE SAMPLING DEFUZZ. VALUE	ELITE SAMPLING METHOD ERROR	ELITE SAMPLING METHOD TIMING
50	0.34%	43	86.00%	0.29%	0.6167008518	0.0146609859	0.0216 secs.
100	0.69%	80	80.00%	0.55%	0.6228289822	0.0085328555	0.0432 secs.
250	1.71%	147	58.80%	1.01%	0.6295092983	0.0018525394	0.108 secs.
500	3.43%	221	44.20%	1.52%	0.6279777881	0.0033840496 ◁	0.215 secs.
750	5.14%	252	33.60%	1.73%	0.6295741240	0.0017877137 ◁	0.328 secs.
1000	6.86%	278	27.80%	1.91%	0.6293312286	0.0020306091 ◁	0.432 secs.
5000	34.29%	408	8.16%	2.80%	0.6306797744	**0.0006820633** ◁	2.16 secs.
10000	68.59%	438	4.38%	3.00%	0.6310563647	**0.0003054730** ◁	5.01 secs.
50000	342.94%	486	0.97%	3.33%	0.6311913249	**0.0001705128** ◁	22.8 secs.
100000	685.87%	486	0.49%	3.33%	0.6313618377	**0.0000000000** ◁	42.9 secs.

Table A.8: Elite sampling results for the HeaterFIS0.125 test set. Number of embedded sets = 14580. Percentage of embedded sets sampled = $\frac{\text{sample size}}{\text{number of embedded sets}} \times 100$. Exhaustive defuzzified value = 0.6313618377. Errors shown in bold are smaller than the VSCTR error. Errors marked '◁' are lower than the corresponding errors for the sampling method.

NO. OF α-PLANES	α-PLANES/ CORL DEFUZZ. VALUE	α-PLANES/ CORL ERROR	α-PLANES/ CORL TIMING	α-PLANES/ INT. EXH. DEFUZZ. VALUE	α-PLANES/ INTERVAL EXHAUSTIVE ERROR
3	0.5974411770	-0.0339206607	0.000566 secs.	0.5974395543	-0.0339222834
5	0.6014928819	-0.0298689558	0.000804 secs.	0.6014844463	-0.0298773914
9	0.6220020252	-0.0093598125	0.00154 secs.	0.6219954766	-0.0093663611
11	0.6202108548	-0.0111509829	0.00178 secs.	0.6202019617	-0.0111598760
21	0.6176529687	-0.0137088690	0.00346 secs.	0.6176441546	-0.0137176831
51	0.6149638697	-0.0163979680	0.00851 secs.	0.6149552604	-0.0164065773
101	0.6146818722	-0.0166799655	0.0169 secs.	0.6146732228	-0.0166886149
1001	0.6149166283	-0.0164452094	0.166 secs.	0.6149079069	-0.0164539308
10001	0.6149818425	-0.0163799952	1.77 secs.	0.6149731309	-0.0163887068
100001	0.6149818643	-0.0163799734	59.6 secs.	0.6149731532	-0.0163886845

Table A.9: α-planes/CORL and α-planes/interval exhaustive results for the HeaterFIS0.125 test set. Exhaustive defuzzified value = 0.6313618377. Error = α-planes value - exhaustive value.

Appendix B. Generalised Test Set Heater0.0625

EXHAUSTIVE DEFUZZIFIED VALUE	NO. OF EMB. SETS	NO. OF NRESS	EXHAUST-IVE TIMING	VSCTR DEFUZZIFIED VALUE	VSCTR ERROR	VSCTR TIMING
0.2621587894	13778100	2774	25.1 mins.	0.2592117473	0.0029470421	0.000453 secs.

Table B.10: Exhaustive and VSCTR results for the HeaterFIS0.0625 test set.

SAMPLE SIZE	PERCENT. OF EMB. SETS SAMPLED	SAMPLING DEFUZZIFIED VALUE	SAMPLING METHOD ERROR	SAMPLING METHOD TIMING
50	0.0004%	0.2634998330	**0.0013410436** ◁	0.0306 secs.
100	0.0007%	0.2643678735	**0.0022090841** ◁	0.0609 secs.
250	0.0018%	0.2639954015	**0.0018366121** ◁	0.152 secs.
500	0.0036%	0.2644544522	**0.0022956628** ◁	0.305 secs.
750	0.0054%	0.2641746630	**0.0020158736** ◁	0.458 secs.
1000	0.0073%	0.2646109558	**0.0024521664** ◁	0.609 secs.
5000	0.0363%	0.2645765948	**0.0024178054**	3.11 secs.
10000	0.0726%	0.2645380675	**0.0023792781**	6.38 secs.
50000	0.3629%	0.2644304187	**0.0022716293**	51.9 secs.
100000	0.7258%	0.2645136689	**0.0023548795**	2.74 mins.

Table B.11: Sampling results for the HeaterFIS0.0625 test set. Number of embedded sets = 13778100. Percentage of embedded sets sampled = $\frac{\text{sample size}}{\text{number of embedded sets}} \times 100$. Exhaustive defuzzified value = 0.2621587894. Errors shown in bold are smaller than the VSCTR error. Errors marked '◁' are lower than the corresponding errors for the elite sampling method.

SAMPLE SIZE	%-AGE OF ESS SAMPLED	NO. OF NRESS IN SAMPLE	NRESS AS %-AGE OF SAMP. SIZE	NRESS AS %-AGE OF ESS	ELITE SAMPLING DEFUZZ. VALUE	ELITE SAMPLING METHOD ERROR	ELITE SAMPLING METHOD TIMING
50	0.0004%	48	96.00%	0.0003%	0.2663946510	0.0042358616	0.0330 secs.
100	0.0007%	94	94.00%	0.0007%	0.2650536879	**0.0028948985**	0.0660 secs.
250	0.0018%	217	86.8o%	0.0016%	0.2654055130	0.0032467236	0.165 secs.
500	0.0036%	358	71.60%	0.0026%	0.2645029187	**0.0023441293**	0.330 secs.
750	0.0054%	491	65.47%	0.0036%	0.2642948052	**0.0021360158**	0.495 secs.
1000	0.0073%	606	60.60%	0.0044%	0.2646455386	**0.0024867492**	0.661 secs.
5000	0.0363%	1140	22.80%	0.0083%	0.2637295835	**0.0015707941** ◁	3.32 secs.
10000	0.0726%	1355	13.55%	0.0098%	0.2635483884	**0.0013895990** ◁	6.67 secs.
50000	0.3629%	1809	3.62%	0.0131%	0.2631277384	**0.0009689490** ◁	33.5 secs.
100000	0.7258%	1958	1.96%	0.0142%	0.2629459523	**0.0007871629** ◁	1.12 mins.

Table B.12: Elite sampling results for the HeaterFIS0.0625 test set. Number of embedded sets = 13778100. Percentage of embedded sets sampled = $\frac{\text{sample size}}{\text{number of embedded sets}} \times 100$. Exhaustive defuzzified value = 0.2621587894. Errors shown in bold are smaller than the VSCTR error. Errors marked '◁' are lower than the corresponding errors for the sampling method.

NO. OF α-PLANES	α-PLANES/ CORL DEFUZZ. VALUE	α-PLANES/ CORL ERROR	α-PLANES/ CORL TIMING	α-PLANES/ INT. EXH. DEFUZZ. VALUE	α-PLANES/ INTERVAL EXHAUSTIVE ERROR
3	0.2911992286	0.0290404392	0.000900 secs.	0.2912056106	0.0290468212
5	0.2843138916	0.0221551022	0.00170 secs.	0.2843202930	0.0221615036
9	0.2781833083	0.0160245189	0.00329 secs.	0.2781887468	0.0160299574
11	0.2791783831	0.0170195937	0.00408 secs.	0.2791839651	0.0170251757
21	0.2839726877	0.0218138983	0.00769 secs.	0.2839784863	0.0218196969
51	0.2845058809	0.0223470915	0.0185 secs.	0.2845118383	0.0223530489
101	0.2857499961	0.0235912067	0.0365 secs.	0.2857559640	0.0235971746
1001	0.2836509843	0.0214921949	0.367 secs.	0.2836568708	0.0214980814
10001	0.2835417182	0.0213829288	3.88 secs.	—	—
100001	0.2835490870	0.0213902976	1.94 mins.	—	—

Table B.13: α-planes/CORL and α-planes/interval exhaustive results for the HeaterFIS0.0625 test set. Exhaustive defuzzified value = 0.2621587894. Error = α-planes value - exhaustive value.

Appendix C. Generalised Test Set Powder0.1

EXHAUSTIVE DEFUZZIFIED VALUE	NO. OF EMB. SETS	NO. OF NRESS	EXHAUST-IVE TIMING	VSCTR DEFUZZIFIED VALUE	VSCTR ERROR	VSCTR TIMING
0.2806983775	24300	1701	2.55 secs.	0.2646964681	0.0160019094	0.000310 secs.

Table C.14: Exhaustive and VSCTR results for the PowderFIS0.1 test set.

SAMPLE SIZE	PERCENT. OF EMB. SETS SAMPLED	SAMPLING DEFUZZIFIED VALUE	SAMPLING METHOD ERROR	SAMPLING METHOD TIMING
50	0.21%	0.2959967354	**0.0152983579**	0.0227 secs.
100	0.41%	0.2983068036	0.0176084261	0.0453 secs.
250	1.03%	0.2879898240	**0.0072914465** ◁	0.113 secs.
500	2.06%	0.2883575902	**0.0076592127** ◁	0.225 secs.
750	3.09%	0.2904003138	**0.0097019363**	0.340 secs.
1000	4.12%	0.2885932629	**0.0078948854**	0.454 secs.
5000	20.58%	0.2893665435	**0.0086681660**	2.32 secs.
10000	41.15%	0.2894760075	**0.0087776300**	4.84 secs.
50000	205.76%	0.2893699018	**0.0086715243**	43.9 secs.
100000	411.52%	0.2896395345	**0.0089411570**	2.46 mins.

Table C.15: Sampling results for the PowderFIS0.1 test set. Number of embedded sets = 24300. Exhaustive defuzzified value = 0.2806983775. Percentage of embedded sets sampled = $\frac{\text{sample size}}{\text{number of embedded sets}} \times 100$. Errors shown in bold are smaller than the VSCTR error. Errors marked '◁' are lower than the corresponding errors for the elite sampling method.

SAMPLE SIZE	%-AGE OF ESS SAMPLED	NO. OF NRESS IN SAMPLE	NRESS AS %-AGE OF SAMP. SIZE	NRESS AS %-AGE OF ESS	ELITE SAMPLING DEFUZZ. VALUE	ELITE SAMPLING METHOD ERROR	ELITE SAMPLING METHOD TIMING
50	0.21%	48	96.00%	0.20%	0.2924967750	**0.0117983975** ◁	0.0254 secs.
100	0.41%	94	94.00%	0.39%	0.2888488173	**0.0081504398** ◁	0.0510 secs.
250	1.03%	199	79.60%	0.82%	0.2886109164	**0.0079125389**	0.127 secs.
500	2.06%	360	72.00%	1.48%	0.2889039270	**0.0082055495**	0.254 secs.
750	3.09%	477	63.60%	1.96%	0.2879727936	**0.0072744161** ◁	0.381 secs.
1000	4.12%	567	56.70%	2.33%	0.2884496935	**0.0077513160** ◁	0.512 secs.
5000	20.58%	1120	22.40%	4.61%	0.2838675377	**0.0031691602** ◁	2.56 secs.
10000	41.15%	1366	13.66%	5.62%	0.2829860213	**0.0022876438** ◁	5.13 secs.
50000	205.76%	1661	3.32%	6.84%	0.2807352287	**0.0000368512** ◁	25.8 secs.
100000	411.52%	1698	1.70%	6.99%	0.2807555453	**0.0000571678** ◁	51.7 secs.

Table C.16: Elite sampling results for the PowderFIS0.1 test set. Number of embedded sets = 24300. Exhaustive defuzzified value = 0.2806983775. Percentage of embedded sets sampled = $\frac{\text{sample size}}{\text{number of embedded sets}} \times 100$. Errors shown in bold are smaller than the VSCTR error. Underlined errors are lower than the errors for the α-planes method, for all numbers of α-planes. Errors marked '◁' are lower than the corresponding errors for the sampling method.

NO. OF α-PLANES	α-PLANES/ CORL DEFUZZ. VALUE	α-PLANES/ CORL ERROR	α-PLANES/ CORL TIMING	α-PLANES/ INT. EXH. DEFUZZ. VALUE	α-PLANES/ INTERVAL EXHAUSTIVE ERROR
3	0.3100683482	0.0293699707	0.000653 secs.	0.3100714646	0.0293730871
5	0.2990422650	0.0183438875	0.00123 secs.	0.2990446820	0.0183463045
9	0.2949801671	**0.0142817896**	0.00238 secs.	0.2949820128	0.0142836353
11	0.2860659413	**0.0053675638** *	0.00296 secs.	0.2860677799	0.0053694024
21	0.2903153362	**0.0096169587**	0.00557 secs.	0.2903173044	0.0096189269
51	0.2928824383	**0.0121840608**	0.0133 secs.	0.2928844669	0.0121860894
101	0.2909066603	**0.0102082828**	0.0267 secs.	0.2909086286	0.0102102511
1001	0.2907821474	**0.0100837699**	0.267 secs.	0.2907840999	0.0100857224
10001	0.2907215619	**0.0100231844**	2.88 secs.	0.2907235112	0.0100251337
100001	0.2907192214	**0.0100208439**	1.89 mins.	0.2907211701	0.0100227926

Table C.17: α-planes/CORL and α-planes/interval exhaustive results for the PowderFIS0.1 test set. Exhaustive defuzzified value = 0.2806983775. Error = α-planes value - exhaustive value. Errors shown in bold are smaller than the VSCTR error. The error marked '*' is lower than every error for the sampling method.

Appendix D. Generalised Test Set Powder0.05

EXHAUSTIVE DEFUZZIFIED VALUE	NO. OF EMB. SETS	NO. OF NRESS	EXHAUST-IVE TIMING	VSCTR DEFUZZIFIED VALUE	VSCTR ERROR	VSCTR TIMING
0.8180632180	3840000	5093	8.22 mins.	0.8185912163	0.0005279983	0.000555 secs.

Table D.18: Exhaustive and VSCTR results for the PowderFIS0.05 test set.

SAMPLE SIZE	PERCENT. OF EMB. SETS SAMPLED	SAMPLING DEFUZZIFIED VALUE	SAMPLING METHOD ERROR	SAMPLING METHOD TIMING
50	0.001%	0.8165757956	0.0014874224 $\lhd$	0.0326 secs.
100	0.003%	0.8173514791	0.0007117389 $\lhd$	0.0648 secs.
250	0.007%	0.8176368830	**0.0004263350** $\lhd$	0.162 secs.
500	0.013%	0.8166109316	0.0014522864	0.323 secs.
750	0.020%	0.8166335918	0.0014296262	0.485 secs.
1000	0.026%	0.8165791599	0.0014840581	0.647 secs.
5000	0.130%	0.8171269807	0.0009362373	3.30 secs.
10000	0.260%	0.8169971802	0.0010660378	6.73 secs.
50000	1.302%	0.8168484040	0.0012148140	54.4 secs.
100000	2.604%	0.8168981632	0.0011650548	2.82 mins.

Table D.19: Sampling results for the PowderFIS0.05 test set. Number of embedded sets = 3840000. Exhaustive defuzzified value = 0.8180632180. Percentage of embedded sets sampled = $\frac{\text{sample size}}{\text{number of embedded sets}} \times 100$. Error shown in bold is smaller than the VSCTR error. Errors marked '$\lhd$' are lower than the corresponding errors for the elite sampling method.

SAMPLE SIZE	%-AGE OF ESS SAMPLED	NO. OF NRESS IN SAMPLE	NRESS AS %-AGE OF SAMP. SIZE	NRESS AS %-AGE OF ESS	ELITE SAMPLING DEFUZZ. VALUE	ELITE SAMPLING METHOD ERROR	ELITE SAMPLING METHOD TIMING
50	0.001%	50	100.00%	0.0013%	0.8163161308	0.0017470872	0.0355 secs.
100	0.003%	94	94.00%	0.0024%	0.8164985098	0.0015647082	0.0709 secs.
250	0.007%	216	86.40%	0.0056%	0.8168936251	0.0011695929	0.178 secs.
500	0.013%	406	81.20%	0.0106%	0.8169408395	0.0011223785 ◁	0.355 secs.
750	0.020%	550	73.33%	0.0143%	0.8168162196	0.0012469984 ◁	0.533 secs.
1000	0.026%	673	67.30%	0.0175%	0.8170654905	0.0009977275 ◁	0.711 secs.
5000	0.130%	1595	31.90%	0.0415%	0.8171645726	0.0008986454 ◁	3.59 secs.
10000	0.260%	2029	20.29%	0.0528%	0.8173314906	0.0007317274 ◁	7.21 secs.
50000	1.302%	3026	6.05%	0.0788%	0.8175959636	**0.0004672544** ◁	36.3 secs.
100000	2.604%	3439	3.44%	0.0896%	0.8177743683	**0.0002888497** ◁	1.22 mins.

Table D.20: Elite sampling results for the PowderFIS0.05 test set. Number of embedded sets = 3840000. Percentage of embedded sets sampled = $\frac{\text{sample size}}{\text{number of embedded sets}} \times 100$. Exhaustive defuzzified value = 0.8180632180. Errors shown in bold are smaller than the VSCTR error. Errors marked '◁' are lower than the corresponding errors for the sampling method.

NO. OF α-PLANES	α-PLANES/ CORL DEFUZZ. VALUE	α-PLANES/ CORL ERROR	α-PLANES/ CORL TIMING	α-PLANES/ INT. EXH. DEFUZZ. VALUE	α-PLANES/ INTERVAL EXHAUSTIVE ERROR
3	0.8371816462	0.0191184282	0.00148 secs.	0.8371808680	0.0191176500
5	0.8132243650	-0.0048388530	0.00244 secs.	0.8132227384	-0.0048404796
9	0.8003904509	-0.0176727671	0.00433 secs.	0.8003883556	-0.0176748624
11	0.8028981616	-0.0151650564	0.00529 secs.	0.8028960507	-0.0151671673
21	0.8000431818	-0.0180200362	0.0101 secs.	0.8000408574	-0.0180223606
51	0.7987563133	-0.0193069047	0.0243 secs.	0.7987538800	-0.0193093380
101	0.7983826038	-0.0196806142	0.0483 secs.	0.7983801575	-0.0196830605
1001	0.7974846584	-0.0205785596	0.479 secs.	0.7974821984	-0.0205810196
10001	0.7974345629	-0.0206286551	50.5 secs.	—	—
100001	0.7974291278	-0.0206340902	2.46 mins.	—	—

Table D.21: α-planes/CORL and α-planes/interval exhaustive results for the PowderFIS0.05 test set. Exhaustive defuzzified value = 0.8180632180. Error = α-planes value - exhaustive value.

Appendix E. Generalised Test Set Shopping0.1

EXHAUSTIVE DEFUZZIFIED VALUE	NO. OF EMB. SETS	NO. OF NRESS	EXHAUST-IVE TIMING	VSCTR DEFUZZIFIED VALUE	VSCTR ERROR	VSCTR TIMING
0.5954109472	312500	2495	32.9 secs.	0.5939161160	0.0014948312	0.000315 secs.

Table E.22: Exhaustive and VSCTR results for the ShoppingFIS0.1 test set.

SAMPLE SIZE	PERCENT. OF EMB. SETS SAMPLED	SAMPLING DEFUZZIFIED VALUE	SAMPLING METHOD ERROR	SAMPLING METHOD TIMING
50	0.02%	0.5893874958	0.0060234514	0.0218 secs.
100	0.03%	0.5905449544	0.0048659928	0.0434 secs.
250	0.08%	0.5926005506	0.0028103966 ◁	0.108 secs.
500	0.16%	0.5926817464	0.0027292008	0.218 secs.
750	0.24%	0.5923095537	0.0031013935	0.325 secs.
1000	0.32%	0.5934992219	0.0019117253	0.435 secs.
5000	1.60%	0.5931185649	0.0022923823	2.23 secs.
10000	3.20%	0.5929055726	0.0025053746	4.60 secs.
50000	16.00%	0.5933037587	0.0021071885	42.4 secs.
100000	32.00%	0.5933184632	0.0020924840	2.43 mins.

Table E.23: Sampling results for the ShoppingFIS0.1 test set. Number of embedded sets = 312500. Percentage of embedded sets sampled = $\frac{\text{sample size}}{\text{number of embedded sets}} \times 100$. Exhaustive defuzzified value = 0.5954109472. Errors marked '◁' are lower than the corresponding errors for the elite sampling method.

SAMPLE SIZE	%-AGE OF ESS SAMPLED	NO. OF NRESS IN SAMPLE	NRESS AS %-AGE OF SAMP. SIZE	NRESS AS %-AGE OF ESS	ELITE SAMPLING DEFUZZ. VALUE	ELITE SAMPLING METHOD ERROR	ELITE SAMPLING METHOD TIMING
50	0.02%	50	100.00%	0.016%	0.5954284597	**<u>0.0000175125</u>** ◁	0.0243 secs.
100	0.03%	94	94.00%	0.030%	0.5935585538	0.0018523934 ◁	0.0485 secs.
250	0.08%	210	84.00%	0.067%	0.5911745884	0.0042363588	0.121 secs.
500	0.16%	368	73.60%	0.118%	0.5954734998	**<u>0.0000625526</u>** ◁	0.243 secs.
750	0.24%	481	64.13%	0.154%	0.5935433933	0.0018675539 ◁	0.366 secs.
1000	0.32%	570	57.00%	0.182%	0.5935158606	0.0018950866 ◁	0.486 secs.
5000	1.60%	1134	22.68%	0.363%	0.5937003262	0.0017106210 ◁	2.45 secs.
10000	3.20%	1401	14.01%	0.448%	0.5948734695	**<u>0.0005374777</u>** ◁	4.92 secs.
50000	16.00%	1943	3.89%	0.622%	0.5949611026	**<u>0.0004498446</u>** ◁	24.8 secs.
100000	32.00%	2146	2.15%	0.687%	0.5952072004	**<u>0.0002037468</u>** ◁	49.9 secs.

Table E.24: Elite sampling results for the ShoppingFIS0.1 test set. Number of embedded sets = 312500. Exhaustive defuzzified value = 0.5954109472. Percentage of embedded sets sampled = $\frac{\text{sample size}}{\text{number of embedded sets}} \times 100$. Errors shown in bold are smaller than the VSCTR error. Underlined errors are lower than the errors for the α-planes method, for all numbers of α-planes. Errors marked '◁' are lower than the corresponding errors for the sampling method.

NO. OF α-PLANES	α-PLANES/ CORL DEFUZZ. VALUE	α-PLANES/ CORL ERROR	α-PLANES/ CORL TIMING	α-PLANES/ INT. EXH. DEFUZZ. VALUE	α-PLANES/ INTERVAL EXHAUSTIVE ERROR
3	0.6151869952	0.0197760480	0.000911 secs.	0.6151852147	0.0197742675
5	0.6018755341	0.0064645869	0.00148 secs.	0.6018735720	0.0064626248
9	0.5932602572	-0.0021506900	0.00261 secs.	0.5932572151	-0.0021537321
11	0.5946487587	**-0.0007621885**	0.00322 secs.	0.5946460014	-0.0007649458
21	0.5929872008	-0.0024237464	0.00608 secs.	0.5929838018	-0.0024271454
51	0.5920148105	-0.0033961367	0.0146 secs.	0.5920110566	-0.0033998906
101	0.5919492352	-0.0034617120	0.0289 secs.	0.5919454769	-0.0034654703
1001	0.5914403564	-0.0039705908	0.288 secs.	0.5914366015	-0.0039743457
10001	0.5914134660	-0.0039974812	3.12 secs.	0.5914097097	-0.0040012375
100001	0.5914058776	-0.0040050696	2.13 mins.	0.5914021206	-0.0040088266

Table E.25: α-planes/CORL and α-planes/interval exhaustive results for the ShoppingFIS0.1 test set. Exhaustive defuzzified value = 0.5954109472. Error = α-planes value - exhaustive value. Error shown in bold is smaller than the VSCTR error.

Appendix F. Generalised Test Set Shopping0.05

EXHAUSTIVE DEFUZZIFIED VALUE	NO. OF EMB. SETS	NO. OF NRESS	EXHAUST-IVE TIMING	VSCTR DEFUZZIFIED VALUE	VSCTR ERROR	VSCTR TIMING
0.1821425020	3840000	12347	11.8 mins.	0.1814087837	0.0007337183	0.000552 secs.

Table F.26: Exhaustive and VSCTR results for the ShoppingFIS0.05 test set.

SAMPLE SIZE	PERCENT. OF EMB. SETS SAMPLED	SAMPLING DEFUZZIFIED VALUE	SAMPLING METHOD ERROR	SAMPLING METHOD TIMING
50	0.001%	0.1826430434	**0.0005005414** ◁	0.0330 secs.
100	0.003%	0.1820101587	**0.0001323433** ◁	0.0656 secs.
250	0.007%	0.1838659287	0.0017234267	0.164 secs.
500	0.013%	0.1831044751	0.0009619731 ◁	0.329 secs.
750	0.020%	0.1829725179	0.0008300159 ◁	0.492 secs.
1000	0.026%	0.1827985154	**0.0006560134** ◁	0.655 secs.
5000	0.130%	0.1830080344	0.0008655324	3.33 secs.
10000	0.260%	0.1831606564	0.0010181544	6.79 secs.
50000	1.302%	0.1830777694	0.0009352674	54.6 secs.
100000	2.604%	0.1830956217	0.0009531197	2.85 mins.

Table F.27: Sampling results for the ShoppingFIS0.05 test set. Number of embedded sets = 3840000. Percentage of embedded sets sampled = $\frac{\text{sample size}}{\text{number of embedded sets}} \times 100$. Exhaustive defuzzified value = 0.1821425020. Errors shown in bold are smaller than the VSCTR error. Errors marked '◁' are lower than the corresponding errors for the elite sampling method.

SAMPLE SIZE	%-AGE OF ESS SAMPLED	NO. OF NRESS IN SAMPLE	NRESS AS %-AGE OF SAMP. SIZE	NRESS AS %-AGE OF ESS	ELITE SAMPLING DEFUZZ. VALUE	ELITE SAMPLING METHOD ERROR	ELITE SAMPLING METHOD TIMING
50	0.001%	50	100.00%	0.0013%	0.1810634451	0.0010790569	0.0360 secs.
100	0.003%	97	97.00%	0.0025%	0.1835334781	0.0013909761	0.0720 secs.
250	0.007%	241	96.40%	0.0063%	0.1820180536	**0.0001244484** ◁	0.180 secs.
500	0.013%	455	91.00%	0.0118%	0.1834734689	0.0013309669	0.360 secs.
750	0.020%	668	89.07%	0.0174%	0.1830600422	0.0009175402	0.542 secs.
1000	0.026%	819	81.90%	0.0213%	0.1828033788	**0.0006608768**	0.723 secs.
5000	0.130%	2538	50.76%	0.0661%	0.1828180409	**0.0006755389** ◁	3.68 secs.
10000	0.260%	3520	35.20%	0.0917%	0.1828462673	**0.0007037653** ◁	7.44 secs.
50000	1.302%	6072	12.14%	0.1581%	0.1826417325	**0.0004992305** ◁	38.2 secs.
100000	2.604%	7209	7.21%	0.1877%	0.1825469839	**0.0004044819** ◁	1.29 mins.

Table F.28: Elite sampling results for the ShoppingFIS0.05 test set. Number of embedded sets = 3840000. Percentage of embedded sets sampled = $\frac{\text{sample size}}{\text{number of embedded sets}} \times 100$. Exhaustive defuzzified value = 0.1821425020. Errors shown in bold are smaller than the VSCTR error. Errors marked '◁' are lower than the corresponding errors for the sampling method.

NO. OF α-PLANES	α-PLANES/ CORL DEFUZZ. VALUE	α-PLANES/ CORL ERROR	α-PLANES/ CORL TIMING	α-PLANES/ INT. EXH. DEFUZZ. VALUE	α-PLANES/ INTERVAL EXHAUSTIVE ERROR
3	0.1628183538	-0.0193241482	0.00153 secs.	0.1628191321	-0.0193233699
5	0.1867756350	0.0046331330	0.00248 secs.	0.1867772616	0.0046347596
9	0.1996095491	0.0174670471	0.00442 secs.	0.1996116444	0.0174691424
11	0.1971018384	0.0149593364	0.00542 secs.	0.1971039493	0.0149614473
21	0.1999568182	0.0178143162	0.0103 secs.	0.1999591426	0.0178166406
51	0.2012436867	0.0191011847	0.0248 secs.	0.2012461200	0.0191036180
101	0.2016173962	0.0194748942	0.0496 secs.	0.2016198425	0.0194773405
1001	0.2025153416	0.0203728396	0.488 secs.	0.2025178016	0.0203752996
10001	0.2025654371	0.0204229351	5.13 secs.	—	—
100001	0.2025708722	0.0204283702	2.44 mins.	—	—

Table F.29: α-planes/CORL and α-planes/interval exhaustive results for the ShoppingFIS0.05 test set. Exhaustive defuzzified value = 0.1821425020. Error = α-planes value - exhaustive value.

Appendix G. Wilcoxon Signed Rank Test Results

FIRST METHOD	SECOND METHOD	T	CONCLUSION
VSCTR	sampling	7	cannot reject H_0
VSCTR	elite sampling	9	cannot reject H_0
VSCTR	α-planes/CORL	2	reject H_0; VSCTR more accurate than AP/CORL
sampling	elite sampling	9	cannot reject H_0
sampling	α-planes/CORL	3	cannot reject H_0
elite sampling	α-planes/CORL	2	reject H_0; ES more accurate than AP/CORL

Table G.30: Comparing the errors from the four generalised defuzzification methods using the One-Sided Wilcoxon Signed Rank Test with $\alpha = 0.05$ and $n = 6$. The critical value is 2. For the sampling and elite sampling methods, the sample size is 50.

FIRST METHOD	SECOND METHOD	T	CONCLUSION
VSCTR	sampling	5	cannot reject H_0
VSCTR	elite sampling	7	cannot reject H_0
VSCTR	α-planes/CORL	2	reject H_0; VSCTR more accurate than AP/CORL
sampling	elite sampling	10	cannot reject H_0
sampling	α-planes/CORL	3	cannot reject H_0
elite sampling	α-planes/CORL	0	reject H_0; ES more accurate than AP/CORL

Table G.31: Comparing the errors from the four generalised defuzzification methods using the One-Sided Wilcoxon Signed Rank Test with $\alpha = 0.05$ and $n = 6$. The critical value is 2. For the sampling and elite sampling methods, the sample size is 100.

FIRST METHOD	SECOND METHOD	T	CONCLUSION
VSCTR	sampling	9	cannot reject H_0
VSCTR	elite sampling	9	cannot reject H_0
VSCTR	α-planes/CORL	2	reject H_0; VSCTR more accurate than AP/CORL
sampling	elite sampling	4	cannot reject H_0
sampling	α-planes/CORL	0	reject H_0; S more accurate than AP/CORL
elite sampling	α-planes/CORL	1	reject H_0; ES more accurate than AP/CORL

Table G.32: Comparing the errors from the four generalised defuzzification methods using the One-Sided Wilcoxon Signed Rank Test with $\alpha = 0.05$ and $n = 6$. The critical value is 2. For the sampling and elite sampling methods, the sample size is 250.

FIRST METHOD	SECOND METHOD	T	CONCLUSION
VSCTR	sampling	8	cannot reject H_0
VSCTR	elite sampling	8	cannot reject H_0
VSCTR	α-planes/CORL	2	reject H_0; VSCTR more accurate than AP/CORL
sampling	elite sampling	8	cannot reject H_0
sampling	α-planes/CORL	0	reject H_0; S more accurate than AP/CORL
elite sampling	α-planes/CORL	0	reject H_0; ES more accurate than AP/CORL

Table G.33: Comparing the errors from the four generalised defuzzification methods using the One-Sided Wilcoxon Signed Rank Test with $\alpha = 0.05$ and $n = 6$. The critical value is 2. For the sampling and elite sampling methods, the sample size is 500.

FIRST METHOD	SECOND METHOD	T	CONCLUSION
VSCTR	sampling	9	cannot reject H_0
VSCTR	elite sampling	10	cannot reject H_0
VSCTR	α-planes/CORL	2	reject H_0; VSCTR more accurate than AP/CORL
sampling	elite sampling	3	cannot reject H_0
sampling	α-planes/CORL	0	reject H_0; S more accurate than AP/CORL
elite sampling	α-planes/CORL	0	reject H_0; ES more accurate than AP/CORL

Table G.34: Comparing the errors from the four generalised defuzzification methods using the One-Sided Wilcoxon Signed Rank Test with $\alpha = 0.05$ and $n = 6$. The critical value is 2. For the sampling and elite sampling methods, the sample size is 750.

FIRST METHOD	SECOND METHOD	T	CONCLUSION
VSCTR	sampling	10	cannot reject H_0
VSCTR	elite sampling	10	cannot reject H_0
VSCTR	α-planes/CORL	2	reject H_0; VSCTR more accurate than AP/CORL
sampling	elite sampling	4	cannot reject H_0
sampling	α-planes/CORL	0	reject H_0; S more accurate than AP/CORL
elite sampling	α-planes/CORL	0	reject H_0; ES more accurate than AP/CORL

Table G.35: Comparing the errors from the four generalised defuzzification methods using the One-Sided Wilcoxon Signed Rank Test with $\alpha = 0.05$ and $n = 6$. The critical value is 2. For the sampling and elite sampling methods, the sample size is 1000.

FIRST METHOD	SECOND METHOD	T	CONCLUSION
VSCTR	sampling	9	cannot reject H_0
VSCTR	elite sampling	5	cannot reject H_0
VSCTR	α-planes/CORL	2	reject H_0; VSCTR more accurate than AP/CORL
sampling	elite sampling	0	reject H_0; ES more accurate than S
sampling	α-planes/CORL	0	reject H_0; S more accurate than AP/CORL
elite sampling	α-planes/CORL	0	reject H_0; ES more accurate than AP/CORL

Table G.36: Comparing the errors from the four generalised defuzzification methods using the One-Sided Wilcoxon Signed Rank Test with $\alpha = 0.05$ and $n = 6$. The critical value is 2. For the sampling and elite sampling methods, the sample size is 5000.

FIRST METHOD	SECOND METHOD	T	CONCLUSION
VSCTR	sampling	9	cannot reject H_0
VSCTR	elite sampling	2	reject H_0; ES more accurate than VSCTR
VSCTR	α-planes/CORL	2	reject H_0; VSCTR more accurate than AP/CORL
sampling	elite sampling	0	reject H_0; ES more accurate than S
sampling	α-planes/CORL	0	reject H_0; S more accurate than AP/CORL
elite sampling	α-planes/CORL	0	reject H_0; ES more accurate than AP/CORL

Table G.37: Comparing the errors from the four generalised defuzzification methods using the One-Sided Wilcoxon Signed Rank Test with $\alpha = 0.05$ and $n = 6$. The critical value is 2. For the sampling and elite sampling methods, the sample size is 10000.

FIRST METHOD	SECOND METHOD	T	CONCLUSION
VSCTR	sampling	9	cannot reject H_0
VSCTR	elite sampling	0	reject H_0; ES more accurate than VSCTR
VSCTR	α-planes/CORL	2	reject H_0; VSCTR more accurate than AP/CORL
sampling	elite sampling	0	reject H_0; ES more accurate than S
sampling	α-planes/CORL	0	reject H_0; S more accurate than AP/CORL
elite sampling	α-planes/CORL	0	reject H_0; ES more accurate than AP/CORL

Table G.38: Comparing the errors from the four generalised defuzzification methods using the One-Sided Wilcoxon Signed Rank Test with $\alpha = 0.05$ and $n = 6$. The critical value is 2. For the sampling and elite sampling methods, the sample size is 50000.

FIRST METHOD	SECOND METHOD	T	CONCLUSION
VSCTR	sampling	8	cannot reject H_0
VSCTR	elite sampling	0	reject H_0; ES more accurate than VSCTR
VSCTR	α-planes/CORL	2	reject H_0; VSCTR more accurate than AP/CORL
sampling	elite sampling	0	reject H_0; ES more accurate than S
sampling	α-planes/CORL	0	reject H_0; S more accurate than AP/CORL
elite sampling	α-planes/CORL	0	reject H_0; ES more accurate than AP/CORL

Table G.39: Comparing the errors from the four generalised defuzzification methods using the One-Sided Wilcoxon Signed Rank Test with $\alpha = 0.05$ and $n = 6$. The critical value is 2. For the sampling and elite sampling methods, the sample size is 100000.