An R-fuzzy and Grey Analysis Framework

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***Abstract*—This paper puts forward the notion of an R-fuzzy and grey analysis framework. This is based on our previous works which involved enhancing the R-fuzzy set and the research undertaken on grey analysis, we believe that this newly proposed framework - the R-fuzzy grey analysis framework (RfGAf), to be a viable methodology to adopt when considering uncertainty modelling. It will be shown that the framework is very well suited in application areas involving perception modelling, where group consensus and subjectivity are prevalent. In such domains a single observation can have a multitude of different perspectives, choosing a single fuzzy value as a representative becomes problematic. The fundamental concept of an R-fuzzy set is that it allows for the collective perception of a populous, and also individualised perspectives to be encapsulated within its membership set. The introduction of a significance measure allowed for the quantification of any membership value contained within any generated R-fuzzy set. This in addition provided one the means to infer from the conditional probability of each contained fuzzy membership value. Such is the pairing of the significance measure and the R-fuzzy concept, it replicates in part, the higher order of complex uncertainty which can be garnered using a type-2 fuzzy approach, with the computational ease and objectiveness of a typical type-1 fuzzy set. This paper utilises the use of grey analysis, in particular, the use of the absolute degree of grey incidence for the inspection of the sequence generated when using the significance measure, when quantifying the degree of significance for each contained fuzzy membership value. Using the absolute degree of grey incidence provides a means to measure the metric spaces between sequences, so that perception divergence can be quantified.**

I. Introduction

The foundation of the R-fuzzy grey analysis framework (RfGAf) is based upon the R-fuzzy set, which is a fuzzy and rough set hybrid, where by the uncertain fuzzy membership value is bounded between a lower and upper rough approximation. The R-fuzzy set was proposed by Yang and Hinde in [1]. It was deemed relatively early on in our research that the R-fuzzy set when compared to other more traditional uncertainty models, fared better when considering certain domain applications, namely those associated to perception based domains. As the membership set of an R-fuzzy set is itself a set, more specifically a *rough set*, a greater amount of detail can be encapsulated. When comparing the R-fuzzy model to that of more established uncertainty models, it has several advantages; some of the major draw backs of existing approaches is that a membership value can be lost to an interval or shadow region. In doing so, one is no longer able to ascertain the object's relevance relative to its interval. In certain instances this may not be too much of a concern, but for domains where perception is being modelled, it should always be preferred that each and every membership value be accounted for and have its relevance quantified. For the likes of; Atanassov intuitionistic fuzzy sets [2], where a membership degree and non-membership degree are given. Shadowed sets [3], where the membership value can either belong to the set (1), not to the set (0), or belong to the shadow region $[0, 1]$, to an unknown capacity. Interval-valued fuzzy sets [4], where the use of an interval is used to characterise the object itself. Type-2 fuzzy sets [5], where the secondary grade of membership is a type-1 fuzzy set. These *established* models will not be able to recognise the difference between the values which are contained within their intervals or shadow regions. As an R-fuzzy set makes use of a rough set for its membership set, the lower approximation will contain all fuzzy membership values that have absolutely been agreed upon by all in the consensus. Whereas, the upper approximation will contain all fuzzy membership values that have at least one vote from the populous. It is generally agreed that a generalised type-2 fuzzy set is very well suited in allowing for a greater amount of detail of uncertainty to be captured, but this is computationally expensive, and as such the interval-valued type-2 approach is often used, where any membership value contained in the footprint-of-uncertainty is given a secondary grade membership of 1.

It was remarked in the original paper by Yang and Hinde that if one could quantify the distribution of contained membership values that constitute the membership set of an R-fuzzy set, then one would be able to create a bridge that links itself to that of a generalised type-2 fuzzy set. The introduction of the significance measure proposed by Khuman et al. in [6], [7], [8] does precisely that, it allows for the captured uncertain membership values to be measured. This facilitates a feasible means to express and infer from complex uncertainty without the inherent difficulties often associated with type-2 fuzzy sets. The implementation of the significance measure with an R-fuzzy set allows for each and every fuzzy membership value to be quantified [6], [7], [8]. The results of which can be further investigated using techniques from grey system theory, which formulates the post analysis component of the proposed framework.

Grey theory is yet another approach for handling uncertainty, first proposed by Deng in [9]. The paradigm places particular emphasis on domains associated with small samples and poor information, where the information may be partially known and partially unknown, a common trait of uncertain systems. The purpose of which is to garner an informed and accurate conclusion based on what little, uncertain information is available. This is generally achieved through the processes of generating, excavating and extracting meaningful content. In doing so, the system's operational behaviours and its laws governing its evolution can be accurately described and acutely monitored [10]. The use of sequences in grey modelling is heavily favoured, it is this component and the absolute degree of grey incidence that will provide the post analysis component of the R-fuzzy and grey analysis framework.

Section II will present the preliminaries for R-fuzzy sets and the significance measure, also introduced is the absolute degree of grey incidence. Section III presents the observations, using a worked example to demonstrate the added benefit of using grey techniques for the inspection and further analysis of the results. Section IV will conclude the paper, providing an overall summary.

II. Preliminaries

We first present the definitions for the *approximations*, the bounding component of an R-fuzzy set.

A. Approximation Preliminaries

Definition 1 (*Approximations [11]*): Assume that $\Lambda = (\mathbb{U}, A)$ is an information system and that $B \subseteq A$ and $X \subseteq \mathbb{U}$. Set X can be approximated based on the information contained in B, via the use of a lower and upper approximation set.

The lower approximation contains all observed objects that wholeheartedly belong to the set X with regards to the information contained in B. It is the union of all equivalence classes in $[x]_B$ which are absolutely contained within set X, and is given by:

$$\underline{B}X = \{x \mid [x]_B \subseteq X\} \tag{1}$$

$$\underline{B}(x) = \bigcup_{x \in \mathbb{U}} \{B(x) : B(x) \subseteq X\}$$

The upper approximation contains all observed objects that have a *possible* affinity to the set X with regards to the information contained in B. It is the union of all equivalence classes that have a non-empty intersection with set X, and is given by:

$$\overline{B}X = \{x \mid [x]_B \cap X \neq \emptyset\} \tag{2}$$

$$\overline{B}(x) = \bigcup_{x \in \mathbb{U}} \{B(x) : B(x) \cap X \neq \emptyset\}$$

B. Fuzzy Set Preliminaries

It is beneficial for the reader to be presented with the definitions for both a type-1 and type-2 fuzzy set, as one will soon see how the new framework in part provides a bridge to that of a generalised type-2 fuzzy approach for handling uncertainty.

Definition 2 (*Fuzzy set [12]*): Let $\mathbb{U}$ represent the universe and let A be a set in $\mathbb{U}$ ($A \subseteq \mathbb{U}$). The fuzzy set A is a set of ordered pairs given by the following expression:

$$A = \{\langle x, \mu_A(x)\rangle \mid x \in \mathbb{U}\} \tag{3}$$

$$A = \sum_{x \in \mathbb{U}} \mu(x)/x$$

A type-2 fuzzy set is a logical extension to that of type-1, whereby the addition of a secondary grade of membership is used. The secondary grade itself is a type-1 fuzzy membership, and provides a three-dimensional perspective, allowing for greater encapsulation of uncertainty.

Definition 3 (*Type-2 fuzzy set [5]*): A type-2 fuzzy set $\tilde{A}$ is characterised by a type-2 membership function $\mu_{\tilde{A}}(x,u)$, where $x \in \mathbb{U}$ and $u \in J_x \subseteq [0,1]$. A type-2 fuzzy set is given by the formal expression:

$$\tilde{A} = \{\langle (x,u), \mu_{\tilde{A}}(x,u)\rangle \mid \forall x \in \mathbb{U}, \forall u \in J_x \subseteq [0,1]\} \tag{4}$$

in which $\mu_{\tilde{A}} : \mathbb{U} \times J_x \rightarrow [0,1]$. $\tilde{A}$ can also be expressed as:

$$\tilde{A} = \int_{\forall x \in \mathbb{U}} \int_{\forall u \in J_x \subseteq [0,1]} \mu_{\tilde{A}}(x,u)/(x,u) \tag{5}$$

$$\tilde{A} = \int_{x \in \mathbb{U}} \int_{u \in J_x} \mu_{\tilde{A}}(x,u)/(x,u) \quad J_x \subseteq [0,1] \tag{6}$$

Where $\int\int$ denotes a union over all admissible x and u values. For discrete universes of discourse, $\int$ is replaced by that of $\sum$.

C. R-Fuzzy Set Preliminaries

We now present the concept of an R-fuzzy set, which makes use of the approximations as given in *Definition 1*.

Definition 4 (*R-fuzzy sets [1]*): Let the pair $apr = (J_x, B)$ be an approximation space on a set of values $J_x = \{v_1, v_2, \ldots, v_n\} \subseteq [0,1]$, and let J_x/B denote the set of all equivalence classes of B. Let $\left(\underline{M}_A(x), \overline{M}_A(x)\right)$ be a rough set in apr. The membership set of an R-fuzzy set A is a rough set $\left(\underline{M}_A(x), \overline{M}_A(x)\right)$, where $x \in \mathbb{U}$, given by:

$$A = \{\langle x, \left(\underline{M}_A(x), \overline{M}_A(x)\right)\rangle \mid \forall x \in \mathbb{U}, \underline{M}_A(x) \subseteq \overline{M}_A(x) \subseteq J_x\} \tag{7}$$

$$A = \sum_{x \in \mathbb{U}} \left(\underline{M_A}(x), \overline{M_A}(x)\right)/x$$

Where $\sum$ is the union of all admissible x elements over the universe of discourse. Each $x_i \in \mathbb{U}$ will have an associated description of membership $d(x_i)$, which describes the belongingness of each x_i with regards to the set $A \subseteq \mathbb{U}$. The set C is the available evaluation criteria from which the consensus of the populous is contained. For each pair $((x_i), c_j)$ where $x_i \in \mathbb{U}$ and $c_j \in C$, a subset $M_{cj}(x_i) \subseteq J_x$ is created, given by:

$$M_{cj}(x_i) = \{v \mid v \in J_x, v \xrightarrow{(d(x_i), c_j)} \text{YES}\} \tag{8}$$

The lower approximation for the rough set $M(x_i)$ is given by:

$$\underline{M}(x_i) = \bigcap_j M_{cj}(x_i) \tag{9}$$

The upper approximation for the rough set $M(x_i)$ is given by:

$$\overline{M}(x_i) = \bigcup_j M_{cj}(x_i) \tag{10}$$

Therefore the rough set approximating the membership $d(x_i)$ for x_i is given as:

$$M(x_i) = \left(\bigcap_j M_{cj}(x_i), \bigcup_j M_{cj}(x_i) \right) \tag{11}$$

It was remarked in the original proposal of R-fuzzy sets by Yang and Hinde [1], that the distribution of the membership function once quantified, could then be used to derive a fuzzy set to give type-2 fuzzy sets. We now present the significance measure, the theorems and proofs that describe how an R-fuzzy set and significance measure pairing can be bridged to that of a type-2 fuzzy set.

D. Significance Measure

We now present the significance measure, originally proposed by Khuman et al. in [6], [7], [8].

Definition 5 (*Degree of significance*): Assume that an R-fuzzy set has already been created using the same notation given in *Definition 4*. This also implies that we have a criteria set C, and in turn, have an established fuzzy membership value set J_x. The total number of all generated subsets for a given R-fuzzy set is denoted by $|N|$. The number of subsets that contain the specific membership value one is inspecting is given by S_v. Each value $v \in J_x$ is evaluated by $c_j \in C$, the frequency of which is the number of times v occurred over $|N|$, this results in the degree of significance given by:

$$\gamma_{\bar{A}}\{v\} = \frac{S_v}{|N|} \tag{12}$$

If the returned degree of significance for any given fuzzy membership value is $\gamma_{\bar{A}}\{v\} = 1$, this implies that the value was absolutely agreed upon by all in the criteria set C, meaning that it belongs to the lower approximation:

$$\underline{M}_A = \left\{ \gamma_{\bar{A}}\{v\} = 1 \mid v \in J_x \subseteq [0,1] \right\} \tag{13}$$

For any membership value to be given a $\gamma_{\bar{A}}\{v\} = 1$, one will know that it will also be included in the upper approximation. This is due to that fact that Eq. (7) states that the lower approximation is a subset of the upper approximation $\underline{M}_A(x) \subseteq \overline{M}_A(x)$. Any returned degree of significance greater than 0 will also be included in the upper approximation:

$$\overline{M}_A = \left\{ \gamma_{\bar{A}}\{v\} > 0 \mid v \in J_x \subseteq [0,1] \right\} \tag{14}$$

These interpretations echo the sentiments of fuzzy set theory as presented in *Definition 2*, whereby an element can be described by its membership function such that it returns any real number in the range $[0,1]$. Except instead of representing the belongingness of an object to a particular set, the significance degree returns the measure of significance, with relation to its descriptor $d(x_i)$, based on its conditional probability of distribution. Eq. (12) can be rewritten so that the collected significance degrees constitute a set, given by the following expression:

$$\bar{A} = \left\{ \langle v, \gamma_{\bar{A}}\{v\} \rangle \mid v \in J_x \subseteq [0,1] \right\} \tag{15}$$

Theorem 1: The significance measure described in *Definition 5* is equivalent to a standard type-1 fuzzy set, if it can be described in the same way as presented in *Definition 2*. Whereby its membership function must satisfy the restriction imposed upon it, such that an object is assigned a degree of inclusion either equalling or within the range of $[0,1]$. Also for equivalence to be satisfied, the continuous set representation must be based upon the apex stick heights of the returned degrees of significance for the triggered membership values satisfying the descriptor being inspected.

Proof 1: From *Definition 5* and *Definition 4*, assume that set A is a descriptor for a particular R-fuzzy set. A traditional type-1 fuzzy set is a collection of ordered pairs. The degree of significance for each membership value belonging to a particular R-fuzzy set is quantified by its membership function $\gamma_{\bar{A}}\{v\} : J_x \rightarrow [0,1]$, such that it can be given by the expression:

$$\bar{A} = \left\{ \langle v, \gamma_{\bar{A}}\{v\} \rangle \mid v \in J_x \right\} \qquad (15 \textit{ revisited})$$

Therefore, based on its descriptor the set will contain ordered pairs of membership values and their associated degrees of significance. One can see this expressions is equivalent to the notation given in Eq. (3):

$$A = \left\{ \langle x, \mu_A(x) \rangle \mid x \in \mathbb{U} \right\} \qquad (3 \textit{ revisited})$$

Where an object is provided with a degree of inclusion relative to the set being inspected. Here we have $v \in J_x$ which is the membership set of membership values instead of $x \in \mathbb{U}$. As J_x provides what essentially is the *universe* of discourse, the significance degree measure does indeed act as an equivalent type-1 fuzzy set, when the set is representative of the descriptor the R-fuzzy set was created for.

Theorem 2: An R-fuzzy set A is equivalent to a type-2 fuzzy set as presented in *Definition 3*, only if we consider the probability distribution of the significance degree measure as a fuzzy membership, then an R-fuzzy set is equivalent to a type-2 fuzzy set with discrete secondary membership functions.

Proof 2: From *Definition 3*, we have (x, u) and $\mu_{\tilde{A}}(x, u)$, where (x, u) is indicative of an intersection, and where $\mu_{\tilde{A}}(x, u)$ represents the amplitude, or stick height of objects for said intersection. From *Definition 4*, an R-fuzzy set uses a rough set to describe its membership, as a result we have $(\underline{M}_A, \overline{M}_A)$, where the lower and upper approximations, $\underline{M}$ and $\overline{M}$, respectively, provide the bounds of the set being approximated, which is the descriptor for set A. The degree of significance as presented in *Definition 5*, describes the conditional distribution of triggered membership values for its descriptor, given by Eq. (15). Where the collection of $\gamma_{\bar{A}}\{x\}$ provides the degree of significance of each and every membership value that satisfied the descriptor. As $\mu_{\tilde{A}}(x, u)$ provides one with the amplitude of objects over the 'footprint of uncertainty', $\bar{A}$ provides one with the degree of significance for all triggered membership values satisfying the requirements given by the descriptor. The equivalence is therefore

TABLE I. HUMAN PERCEPTION BASED ON THE VARIATIONS FOR THE COLOUR BLUE

#	*Sex*	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9
p_1	*M*	*DB*	*LB*	*LB*	*DB*	*DB*	*B*	*LB*	*LB*	*B*
p_2	*F*	*DB*	*LB*	*LB*	*B*	*DB*	*B*	*LB*	*LB*	*LB*
p_3	*M*	*DB*	*LB*	*LB*	*DB*	*DB*	*B*	*LB*	*LB*	*LB*
p_4	*M*	*DB*	*LB*	*B*	*DB*	*DB*	*B*	*LB*	*LB*	*LB*
p_5	*M*	*DB*	*LB*	*B*	*DB*	*DB*	*B*	*LB*	*LB*	*LB*
p_6	*M*	*DB*	*LB*	*B*	*DB*	*DB*	*B*	*DB*	*LB*	*LB*
p_7	*F*	*DB*	*LB*	*B*	*DB*	*DB*	*B*	*LB*	*B*	*LB*
p_8	*F*	*DB*	*LB*	*DB*	*DB*	*DB*	*B*	*DB*	*LB*	*LB*
p_9	*M*	*DB*	*LB*	*B*	*B*	*DB*	*B*	*LB*	*LB*	*LB*
p_{10}	*M*	*DB*	*LB*	*B*	*DB*	*DB*	*B*	*LB*	*LB*	*LB*
p_{11}	*M*	*DB*	*LB*	*B*	*DB*	*DB*	*B*	*LB*	*LB*	*LB*
p_{12}	*M*	*DB*	*LB*	*B*	*B*	*DB*	*B*	*LB*	*B*	*LB*
p_{13}	*M*	*DB*	*LB*	*B*	*B*	*DB*	*B*	*LB*	*B*	*LB*
p_{14}	*M*	*DB*	*LB*	*DB*	*DB*	*DB*	*B*	*LB*	*LB*	*LB*
p_{15}	*M*	*DB*	*LB*	*DB*	*DB*	*DB*	*B*	*LB*	*LB*	*LB*
p_{16}	*F*	*DB*	*LB*	*DB*	*DB*	*DB*	*B*	*LB*	*B*	*LB*
p_{17}	*F*	*DB*	*LB*	*B*	*B*	*DB*	*B*	*LB*	*LB*	*LB*
p_{18}	*M*	*DB*	*LB*	*DB*	*DB*	*DB*	*B*	*LB*	*B*	*LB*
p_{19}	*M*	*DB*	*LB*	*DB*	*DB*	*DB*	*B*	*LB*	*LB*	*LB*
p_{20}	*M*	*DB*	*LB*	*B*	*B*	*DB*	*B*	*LB*	*LB*	*LB*

straightforward; both approaches make use of a set, which itself describes the distribution of that set.

E. Grey Theory

The traditional degree of grey incidence provides the basis for all variances of the degree of incidence; $\Gamma = [\gamma_{ij}]$, where each entry in the i^{th} row of the matrix is the degree of grey incidence for the corresponding characteristic sequence Y_i, and relevant behavioural factors $X_1, X_2, \ldots, X_m$. Each entry for the j^{th} column is reference to the degrees of grey incidence for the characteristic sequences $Y_1, Y_2, \ldots, Y_n$ and behavioural factors X_m. The absolute degree of grey incidence $A = [\epsilon_{ij}]_{n\times m}$, is defined as follows:

Definition 6 (*Absolute degree of grey incidence [13][10]*): Assume that X_i and $X_j \in \mathbb{U}$ are two sequences of data with the same magnitude, that are defined as the sum of the distances between two consecutive time points, whose zero starting points have already been computed:

$$s_i = \int_1^n (X_i - x_i(1))dt \qquad (16)$$

$$s_i - s_j = \int_1^n (X_i^0 - X_j^0)dt \qquad (17)$$

Therefore the absolute degree of incidence is given as:

$$\epsilon_{ij} = \frac{1 + |s_i| + |s_j|}{1 + |s_i| + |s_j| + |s_i - s_j|} \qquad (18)$$

This will provide the coefficient values between the contained clusters belonging to the criteria set C. Much like the work the authors did with regards to natural language processing using grey analysis in [14], the absolute degree of grey incidence was utilised to measure the metric spaces of the sequence curves for an optimal string, against the input strings. The returned degree of incidence scored the overall similarity, the higher the value was to 1, the greater the similarity of the two strings. This provides the post-analysis aspect of the framework.

III. OBSERVATIONS

This section will showcase the use of the proposed R-fuzzy grey analysis framework. An example is put forward to further explain the advantages of such a framework. Our previous works have demonstrated the approach used on distribution that did not contain *voids*, this example will demonstrate no matter how volatile the data obtained, the RfGAf approach can still be applied with no loss of information.

Example 1: Assume that $F = \{f_1, f_2, \ldots, f_9\}$ is a set containing 9 different colour swatches based on the colour blue:

$f_1 \rightarrow$ $[25, 25, 112] \rightarrow$

$f_2 \rightarrow$ $[204, 204, 255] \rightarrow$

$f_3 \rightarrow$ $[42, 82, 190] \rightarrow$

$f_4 \rightarrow$ $[0, 51, 153] \rightarrow$

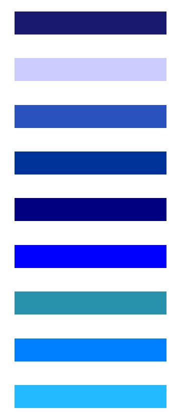
$f_5 \rightarrow$ $[0, 0, 128] \rightarrow$

$f_6 \rightarrow$ $[0, 0, 255] \rightarrow$

$f_7 \rightarrow$ $[40, 146, 172] \rightarrow$

$f_8 \rightarrow$ $[0, 128, 255] \rightarrow$

$f_9 \rightarrow$ $[36, 186, 255] \rightarrow$



The example investigates the collected perception of 20 individuals, with regards to the colour blue. The more observations contained in the criteria set, the greater the chances of disjoint distribution. The more individuals to give their perception, the more likely confliction may arise. These collected perceptions are presented in TABLE I. Notice the inclusion of the *Sex* attribute, of which C contains 15 males and 5 females.

The colours themselves are given by their [RGB] values, from which the average is worked out and stored in N. The values contained are given as $N = \{54, 221, 105, 68, 43, 85, 119, 128, 159\}$. Each average N_i value will correspond to a specific colour swatch F_i. For example, the swatch associated with f_3 has a value of 105, f_5 will be related to 43, and so on. Assume that the criteria set $C = \{\mathrm{p}_1, \mathrm{p}_2, \ldots, \mathrm{p}_{20}\}$ contains the perceptions of 20 individuals, all of whom gave their own opinions based on the available descriptors and the swatches themselves.

The descriptor terms contained within the table can be understood as meaning:

DB $\rightarrow$ Dark Blue   *B* $\rightarrow$ Blue   *LB* $\rightarrow$ Light Blue

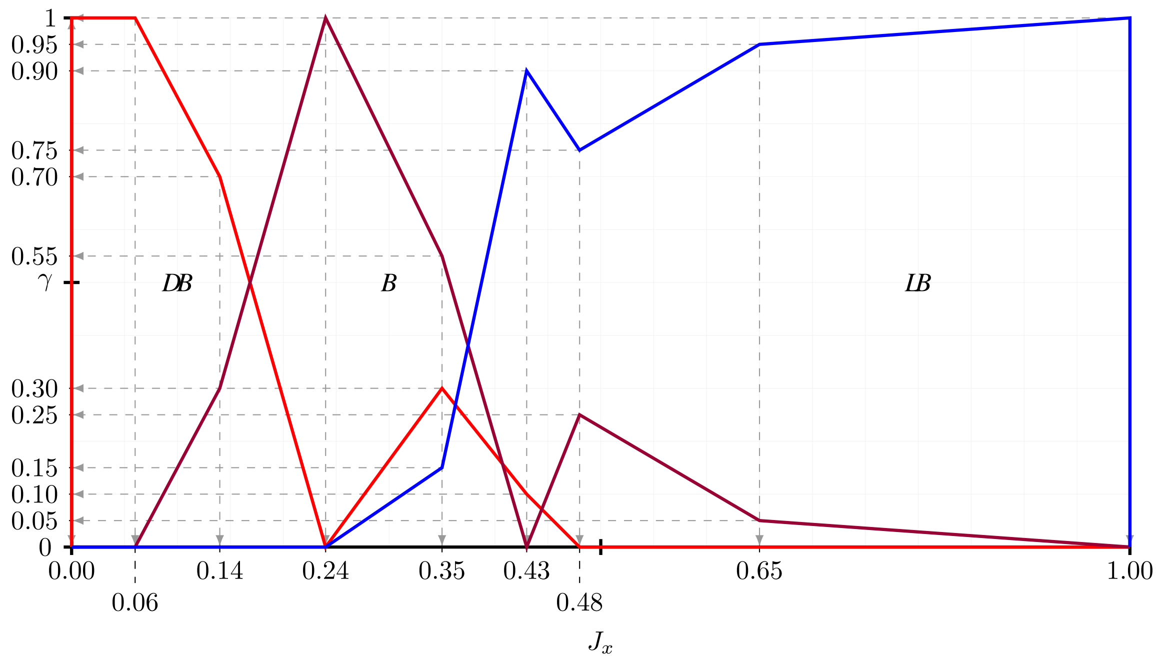


Fig. 1. A derivable continuous visualisation for *Example 1*, based on all the generated significance measures for the R-fuzzy sets of *DB*, *V* & *LB*

The fuzzy membership set J_x is created using a simple linear function:

$$\mu(f_i) = \frac{l_i - l_{min}}{l_{max} - l_{min}} \qquad (19)$$

Using the linear function given in Eq. (19), the resulting fuzzy membership set is given as:

$$J_x = \Big\{0.06, 1.00, 0.35, 0.14, 0.00, 0.24, 0.43, 0.48, 0.65\Big\}$$

Using *Definition 4*, the final generated R-fuzzy sets based on the collected subsets for *LB*, *B* and *DB*, respectively, are given as:

$$DB = \Big(\big\{0.00, 0.06\big\}, \big\{0.00, 0.06, 0.14, 0.35, 0.43\big\}\Big)$$

$$B = \Big(\big\{0.24\big\}, \big\{0.14, 0.24, 0.35, 0.48, 0.65\big\}\Big)$$

$$LB = \Big(\big\{1.00\big\}, \big\{0.35, 0.43, 0.48, 0.65, 1.00\big\}\Big)$$

By using Eq. (12), one is able to calculate the degree of significance for each and every encapsulated fuzzy membership value, from J_x that has an affinity to its R-fuzzy set. The returned degrees of significance for all generated R-fuzzy sets are presented in TABLE II.

Fig. 1 provides one with a derivable continuous representation of the generated R-fuzzy sets and the returned degrees of significance. Given that $\gamma_{\overline{DB}}\{0.24\} = 0.00$, there exists an area of disjointness between 0.14 and 0.35. Referring back to TABLE II, one can clearly see where these areas of disjointness occur and how they are reflected in Fig. 1. The R-fuzzy set *LB* has considerable variance throughout its duration, as can be seen in its fluctuations, it does not however have an area of disjointness. All three generated R-fuzzy sets do have at least one value which returned a significance degree of 1, so even with the extra members for the criteria set, there is still a value that exists indicative of the collective perception held.

Furthermore, as the membership set J_x does indeed remain the same, we can use this as the sequence needed for the absolute degree of grey incidence component. The membership values themselves act as the discretised points along the x axis, whereas the *varying* significance degrees give the associated amplitude. Regardless of how small or large the difference between the returned degrees of significance for comparable fuzzy membership values, the fact that there can be a difference should provide one the motivation to explore further. It is precisely this aspect of wanting to investigate that warrants the use of the absolute degree of grey incidence. Since we now

TABLE II. THE DEGREES OF SIGNIFICANCE BASED ON TABLE I

DB J_x	γ	*B* J_x	γ	*LB* J_x	γ
$\gamma_{\overline{DB}}\{0.00\}$ =	1.00	$\gamma_{\overline{B}}\{0.00\}$ =	0.00	$\gamma_{\overline{LB}}\{0.00\}$ =	0.00
$\gamma_{\overline{DB}}\{0.06\}$ =	1.00	$\gamma_{\overline{B}}\{0.06\}$ =	0.00	$\gamma_{\overline{LB}}\{0.06\}$ =	0.00
$\gamma_{\overline{DB}}\{0.14\}$ =	0.70	$\gamma_{\overline{B}}\{0.14\}$ =	0.30	$\gamma_{\overline{LB}}\{0.14\}$ =	0.00
$\gamma_{\overline{DB}}\{0.24\}$ =	0.00	$\gamma_{\overline{B}}\{0.24\}$ =	1.00	$\gamma_{\overline{LB}}\{0.24\}$ =	0.00
$\gamma_{\overline{DB}}\{0.35\}$ =	0.30	$\gamma_{\overline{B}}\{0.35\}$ =	0.55	$\gamma_{\overline{LB}}\{0.35\}$ =	0.15
$\gamma_{\overline{DB}}\{0.43\}$ =	0.10	$\gamma_{\overline{B}}\{0.43\}$ =	0.00	$\gamma_{\overline{LR}}\{0.43\}$ =	0.90
$\gamma_{\overline{DB}}\{0.48\}$ =	0.00	$\gamma_{\overline{B}}\{0.48\}$ =	0.25	$\gamma_{\overline{LB}}\{0.48\}$ =	0.75
$\gamma_{\overline{DB}}\{0.65\}$ =	0.00	$\gamma_{\overline{B}}\{0.65\}$ =	0.05	$\gamma_{\overline{LB}}\{0.65\}$ =	0.95
$\gamma_{\overline{DB}}\{1.00\}$ =	0.00	$\gamma_{\overline{B}}\{1.00\}$ =	0.00	$\gamma_{\overline{LB}}\{1.00\}$ =	1.00

TABLE III. THE DEGREES OF SIGNIFICANCE FOR MALES

DB J_x	γ	*B* J_x	γ	*LB* J_x	γ
$\gamma_{\overline{DB}}\{0.00\} =$	1.00	$\gamma_{\overline{B}}\{0.00\} =$	0.00	$\gamma_{\overline{LB}}\{0.00\} =$	0.00
$\gamma_{\overline{DB}}\{0.06\} =$	1.00	$\gamma_{\overline{B}}\{0.06\} =$	0.00	$\gamma_{\overline{LB}}\{0.06\} =$	0.00
$\gamma_{\overline{DB}}\{0.14\} =$	0.73	$\gamma_{\overline{B}}\{0.14\} =$	0.27	$\gamma_{\overline{LB}}\{0.14\} =$	0.00
$\gamma_{\overline{DB}}\{0.24\} =$	0.00	$\gamma_{\overline{B}}\{0.24\} =$	1.00	$\gamma_{\overline{LB}}\{0.24\} =$	0.00
$\gamma_{\overline{DB}}\{0.35\} =$	0.27	$\gamma_{\overline{B}}\{0.35\} =$	0.60	$\gamma_{\overline{LB}}\{0.35\} =$	0.13
$\gamma_{\overline{DB}}\{0.43\} =$	0.07	$\gamma_{\overline{B}}\{0.43\} =$	0.00	$\gamma_{\overline{LR}}\{0.43\} =$	0.93
$\gamma_{\overline{DB}}\{0.48\} =$	0.00	$\gamma_{\overline{B}}\{0.48\} =$	0.20	$\gamma_{\overline{LB}}\{0.48\} =$	0.80
$\gamma_{\overline{DB}}\{0.65\} =$	0.00	$\gamma_{\overline{B}}\{0.65\} =$	0.07	$\gamma_{\overline{LB}}\{0.65\} =$	0.93
$\gamma_{\overline{DB}}\{1.00\} =$	0.00	$\gamma_{\overline{B}}\{1.00\} =$	0.00	$\gamma_{\overline{LB}}\{1.00\} =$	1.00

TABLE IV. THE DEGREES OF SIGNIFICANCE FOR FEMALES

DB J_x	γ	*B* J_x	γ	*LB* J_x	γ
$\gamma_{\overline{DB}}\{0.00\} =$	1.00	$\gamma_{\overline{B}}\{0.00\} =$	0.00	$\gamma_{\overline{LB}}\{0.00\} =$	0.00
$\gamma_{\overline{DB}}\{0.06\} =$	1.00	$\gamma_{\overline{B}}\{0.06\} =$	0.00	$\gamma_{\overline{LB}}\{0.06\} =$	0.00
$\gamma_{\overline{DB}}\{0.14\} =$	0.60	$\gamma_{\overline{B}}\{0.14\} =$	0.40	$\gamma_{\overline{LB}}\{0.14\} =$	0.00
$\gamma_{\overline{DB}}\{0.24\} =$	0.00	$\gamma_{\overline{B}}\{0.24\} =$	1.00	$\gamma_{\overline{LB}}\{0.24\} =$	0.00
$\gamma_{\overline{DB}}\{0.35\} =$	0.40	$\gamma_{\overline{B}}\{0.35\} =$	0.40	$\gamma_{\overline{LB}}\{0.35\} =$	0.20
$\gamma_{\overline{DB}}\{0.43\} =$	0.20	$\gamma_{\overline{B}}\{0.43\} =$	0.00	$\gamma_{\overline{LR}}\{0.43\} =$	0.80
$\gamma_{\overline{DB}}\{0.48\} =$	0.00	$\gamma_{\overline{B}}\{0.48\} =$	0.40	$\gamma_{\overline{LB}}\{0.48\} =$	0.60
$\gamma_{\overline{DB}}\{0.65\} =$	0.00	$\gamma_{\overline{B}}\{0.65\} =$	0.00	$\gamma_{\overline{LB}}\{0.65\} =$	1.00
$\gamma_{\overline{DB}}\{1.00\} =$	0.00	$\gamma_{\overline{B}}\{1.00\} =$	0.00	$\gamma_{\overline{LB}}\{1.00\} =$	1.00

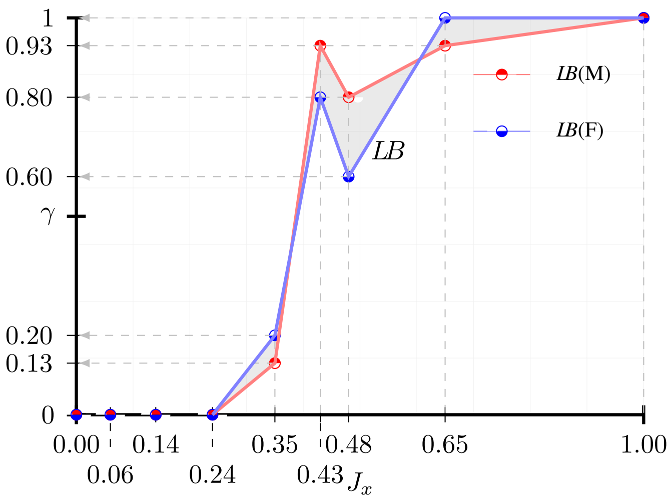


Fig. 2. The Comparability Between Males and Females for *LB*

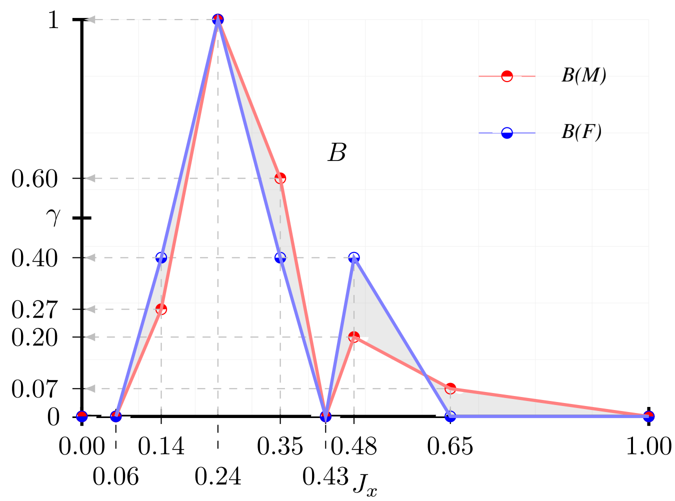


Fig. 3. The Comparability Between Males and Females for *B*

have established sequences indicative of the fuzzy membership set J_x, we can now measure the difference between the metric spaces of comparable sequences based on the returned degrees of significance.

As the data contained in TABLE I contains a *Sex* attribute, one can generate two subsets with relation to male and female. This allows for one to create R-fuzzy subsets, from which the returned degrees of significance can be used to generate the comparable sequences needed, for the absolute degree of grey incidence. Regardless of how the encapsulation of the returned degrees of significance look, it still provides a valid sequence for comparisons to be undertaken, as any generated subset will be indicative of the overall R-fuzzy sets generated from it, along with disjointness and all.

TABLE III and TABLE IV, contain the returned degrees of significance relating to males and females, respectively, all collected from TABLE I. The plots contained in Fig. 2, Fig. 3 and Fig. 4, show the comparable sequences computed from the returned degrees of significance, for *DB*, *B* and *LB*. TABLE V provides a summary of the collected absolute degree of grey

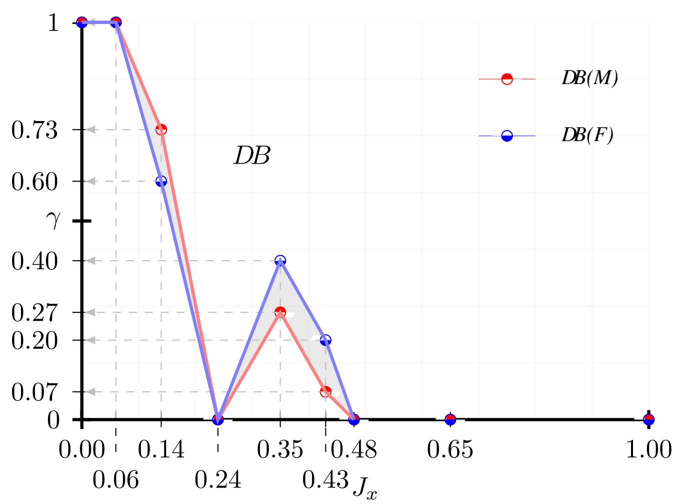


Fig. 4. The Comparability Between Males and Females for *DB*

incidence values for all generated R-fuzzy sets and significance

TABLE V. A COMPARABLE SUMMARY OF THE RETURNED ABSOLUTE DEGREE OF GREY INCIDENCE FOR LB, B & DB

LB	M	F	B	M	F	DB	M	F
M	$\epsilon(1.00)$	$\epsilon(0.940)$	M	$\epsilon(1.00)$	$\epsilon(0.841)$	M	$\epsilon(1.00)$	$\epsilon(0.968)$
F	-	$\epsilon(1.00)$	F	-	$\epsilon(1.00)$	F	-	$\epsilon(1.00)$

measure sequences, for each comparable permutation, of which there are 3. Inspecting the table one can see that sequences generated for *DB* shared the most similarities with a returned metric of $\epsilon(0.968)$. This was then followed by *LB*, with a metric of $\epsilon(0.940)$, therefore the greatest divergence exists for *B*, with a metric of $\epsilon(0.841)$.

IV. CONCLUSION

The R-fuzzy and grey analysis framework is an uncertainty model, one which hybridises that of R-fuzzy, which itself is a fuzzy and rough set pairing; to that of the significance measure and the absolute degree of grey incidence.

Given that an R-fuzzy set allows for the encapsulation of a general consensus and also individual perspectives, the wealth of information an R-fuzzy set can contain is a great deal. The introduction of the significance measure by the authors in [6], [7], [8] has allowed for the R-fuzzy concept to model more complex uncertainty, returning a higher dimension of results for better inferencing. With the introduction of grey analysis, specifically the use of the absolute degree of grey incidence, it has been shown that even more information and inference can be garnered from the same initial data set.

With the increase in observers contained within the criteria set, comes a greater chance of disjointness. The sequences generated for *Example 1* contained areas of disjointness and volatility, as indicated by the plots. The sequences themselves often criss-cross with one another. This is not a problem, for the absolute degree of grey incidence uses the absolute values for the area encapsulated between discretised points. As such, a reliable and detailed metric can be returned from which an insight provides the perceived perceptions of males and females. As there were 3 times more males as compared to females, the overall similarities between sequences as indicated by the high returned metrics, would indicate that with more females, the value would only increase slightly, if at all. The robustness that the R-fuzzy grey analysis framework has, allows for it to be executed on clusters with uneven frequency, with relatively small amounts of data. This enables whatever permutations are contained within the criteria set, to be able to be compared and contrasted. As the comparisons are of the same R-fuzzy descriptors, the general overall similarity will be prevalent, however, the enhanced R-fuzzy approach is best utilised to provide a metric for the divergences that *could* exist.

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