Aggregation of Unbalanced Fuzzy Linguistic Information in Decision Problems based on Type-1 OWA Operator

Francisco Mata*, Luis G. Pérez*, Francisco Chiclana† and Enrique Herrera-Viedma‡
*Department of Computer Science, University of Jaén, 23700 Jaén, Spain, fmata@ujaen.es, lgonzaga@ujaen.es
†Centre for Computational Intelligence, De Montfort University, Leicester, LE1 9BH, UK, chiclana@dmu.ac.uk
‡ Department of Computer Science, University of Granada, Granada, Spain, viedma@decsai.ugr.es

***Abstract*—Information aggregation is a key task in any group decision making problem. In the fuzzy linguistic context, when comparing two alternatives, it is usually assumed that assessments belong to linguistic term sets of symmetrically distributed labels with respect to a central label that stands for the indifference state. However, in practice there are many situations whose nature recommends their modelling using not symmetric linguistic term sets, and therefore formal approaches to deal with sets of unbalanced linguistic labels in decision making are necessary to be appropriately developed. In literature, the linguistic hierarchy methodology has proved successful when modelling unbalanced linguistic labels using an ordinal approach in their representation. However, linguistic labels can be modelled using a cardinal approach, i.e. as fuzzy subsets represented by membership functions. Obviously, the linguistic hierarchy methodology is not appropriate in these cases. In this contribution, a Type-1 OWA approach is proposed to deal with the aggregation step of the resolution process of a group decision making problem with unbalanced linguistic information modelled using a cardinal approach. The Type-1 OWA operator aggregates fuzzy sets and uses whole membership functions to compute the aggregated output fuzzy sets. The application of the Type-1 OWA approach to an example where the linguistic hierarchy approach was applied before will provide us an opportunity to compare the aggregated results obtained in both cases. Following the defuzzification of the Type-1 OWA aggregated values, it can be concluded that both methodologies are equivalent. The use of the Type-1 OWA approach in this decision making context does not require building linguistic hierarchies while at the same time allows a fully exploitation of the fuzzy nature of linguistic information.**

I. Introduction

Fuzzy Linguistic Approach [1] has proved successful in manifold research areas in which imprecision and vagueness associated to qualitative information is present. Key elements of this approach are the concept of linguistic variable and the semantic rules to associate to each element of the universe of discourse its meaning. The association among linguistic variable and its semantic has been resolved through fuzzy sets. Another important aspect related to linguistic approach is the granularity of the uncertainty, i.e. the grade of distinction or discrimination of the uncertainty that is represented by the cardinality of the corresponding linguistic term set [2].

Most decision making problems with linguistic information are modelled assuming linguistic term sets with a symmetric and uniform distribution of the linguistic assessments on the universe of discourse. Clearly this approach may be appropriate to problems where the distinction of uncertainty is proportional and equal among the set of linguistic terms, but there exist problems where this does not happen. For example if we consider the (educational) grading system depicted in Fig. 1, we observe as in the right side of the scale there are more terms than in the left side, and that the triangular fuzzy sets representing the semantic of each assessment are different. This is a typical case where dealing with unbalanced linguistic term sets is required [3], [4].

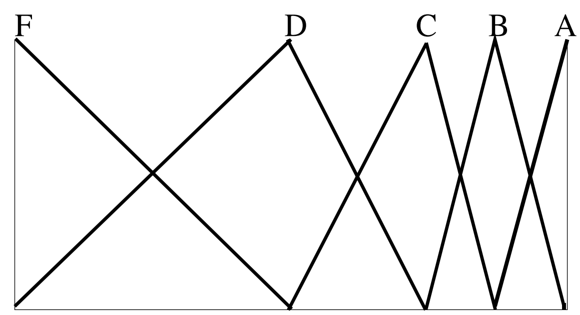

Fig. 1. Semantic representation of the granding system

In literature, there exist several methodologies to address decision making problems with unbalanced linguistic term sets based on the 2-tuple information representation model [7], [8], although in this contribution we focus on so-called linguistic hierarchy methodology (LH). LH was introduced in [5] and later applied in [6] to improve the precision in processes of computing with words in multi-granular linguistic contexts. In [4], it was successfully demonstrated how this methodology could be applied in aggregating unbalanced linguistic information. In summary, LH consists of building a structure with several levels where each one represents a linguistic term set with different granularity to the remaining levels of the hierarchy, maintaining the former modal points and achieving a smooth transitions between successive levels. By means of transformation functions, linguistic labels of a level can be transformed to labels of different levels without loss of information. In this way all unbalanced linguistic term sets can be mapped with its appropriate linguistic term within the structure, and are transformed to a common domain with maximum granularity which ultimately are aggregated using the 2-tuple linguistic computational model. This methodology, which has later been improved regarding the construction of the LH [9], is complex to put it into practice.

A different approach to tackle problems with unbalanced linguistic information is via the Type-1 Ordered Weighted Average (Type-1 OWA) operator [10]. This operator has been developed via the application of the extension principle to Yager's OWA operator [11]. Among the set of features that define this operator, we would highlight two that we deem crucial for this framework. The first one is its capacity to aggregate directly linguistic terms even when they are modelled using different type of membership functions (triangular, trapezoidal, etc.), within the context of balanced or unbalanced linguistic context. The second one refers to the capability of the Type-1 OWA to make use of the whole membership function of the fuzzy sets to aggregate. Both characteristic allow us to conclude that it is a fitting aggregation operator to operate with fuzzy unbalanced linguistic information. Type-1 OWA has been successfully applied to aggregate linguistic information with linguistic weights [12], [13] and to address consensus reaching processes with multi-granular linguistic information [14].

The aim of this contribution is to introduce the Type-1 OWA operator as a precise, reliable and useful operator to aggregate unbalance fuzzy linguistic information in GDM problems. This operator is able to aggregate linguistic term sets with different semantic and granularity when they are represented as fuzzy sets. Notice that in [15], the representation of linguistic assessments using the cardinal approach based on the use of fuzzy sets, and the ordinal approach based on the use of the 2–tuples were proved to be mathematically isomorphic when fuzzy numbers are ranked using their respective centroids. Thus, the Type-1 OWA methodology put forward here may be seen as an extension of the linguistic symbolic computational model based on ordinal scales and indexes.

This contribution is structured as follows. Section II revises the the 2-tuples information representation model (II-A) as well as the LH methodology (II-B). Section III includes a detailed description of Type-1 OWA operator and its application in aggregating unbalanced fuzzy linguistic information using the same data used in [4] where the LH methodology was applied. Following a comparison of the results obtained using the Type-1 OWA approach and the previously reported using the LH approach is provided. The paper is closed with Section IV where conclusions are drawn.

II. Preliminaries

This section presents a brief review of the basic concepts concerning the 2-tuples linguistic representation model and fundamentals of linguistic hierarchies as used in unbalanced linguistic decision contexts.

A. The 2-Tuple Linguistic Representation Model

The 2-tuple linguistic representation model was presented in [16] and it is the basis of the computational model for the LH methodology,

This linguistic model takes as a basis the symbolic representation model based on indexes and in addition defines the concept of symbolic translation to represent the linguistic information by means of a pair of values called linguistic 2–tuple, (s_i, α_i), where s_i is one of the original linguistic terms (i.e. $s_i \in S = \{s_0, ..., s_g\}$) and $\alpha_i \in [-.5, .5)$ is a numeric value representing the symbolic translation. This representation structure allows, on the one hand, to obtain the same information than with the symbolic representation model based on indexes without losing information in the aggregation phase. On the other hand, the result of the aggregation is expressed on the same domain as the initial linguistic labels.

Definition 1 (Linguistic 2–tuple representation): Let $\beta \in [0, g]$ be the result of a symbolic aggregation of the indexes of a set of labels assessed in a linguistic term set $S = \{s_0, ..., s_g\}$. Let $i = round(\beta) \in \{0, \ldots, g\}$. The value $\alpha_i = \beta - i \in [-0.5, 0.5)$ is called a *symbolic translation*, and the pair of values (s_i, α_i) is called the 2*–tuple linguistic representation model.*

This model defines a set of functions between linguistic 2-tuples and numerical values.

Definition 2: Let $S = \{s_0, \ldots, s_g\}$ be a set of linguistic terms. The *2-tuple set associated with S* is defined as $\langle S \rangle = S \times [-0.5, 0.5)$. We define the function $\Delta : [0, g] \longrightarrow \langle S \rangle$ given by,

$$\Delta(\beta) = (s_i, \alpha), \quad \text{with} \quad \begin{cases} i = \text{ round}(\beta), \\ \alpha = \beta - i, \end{cases}$$

where *round* assigns to β the integer number $i \in \{0, 1, \ldots, g\}$ closest to β.

We note that Δ is bijective [16] and $\Delta^{-1} : \langle S \rangle \longrightarrow [0, g]$ is defined by $\Delta^{-1}(s_i, \alpha) = i + \alpha$. In this way, the 2-tuples of $\langle S \rangle$ will be identified with the numerical values in the interval $[0, g]$. This representation model has associated a computational model that was presented in [16].

B. Linguistic Hierarchies

The LH was introduced in [5] as a methodology to compute with words without loss information, which was subsequently improved in [6].

A LH is a set of levels, where each level consists of a linguistic term set with different granularity to the remaining levels of the hierarchy. Each level of a LH is denoted as *l(t,n(t))*, with *t* indicating the level of the hierarchy and *n(t)* the granularity of the linguistic term set of level *t*. It is assumed that all levels of a LH contain linguistic term sets with an odd number of terms, modelled via triangular membership functions and symmetrical and uniformly distributed in $[0, 1]$. The levels belonging to a LH are ordered according to their granularity. A LH is also defined as the union of all its levels t: $LH = \bigcup_t l(t, n(t))$.

1) Building linguistic hierarchies: Given a LH, with linguistic term set at level t, $S = \{s_0, ..., s_{n(t)-1}\}$, $s_k \in S, (k = 0, ..., n(t) - 1)$, and $n(t)$ its granularity of uncertainty, then this is denoted as follows:

$$S^{n(t)} = \{s_0^{n(t)}, ..., s_{n(t)-1}^{n(t)}\}.$$

The methodology to construct a LH increases the granularity of each linguistic term set at each level, and it was presented in [6]. It consists of the following rules, so-called *linguistic hierarchy basic rules*:

1) To preserve all *former modal points* of the membership functions of each linguistic term from one level to the following one.
2) To make *smooth transitions between successive levels* in order to build a new linguistic term set, $S^{n(t+1)}$. The new linguistic term will be added between each pair of terms belonging to the term set of the previous level t (see Fig. 2).

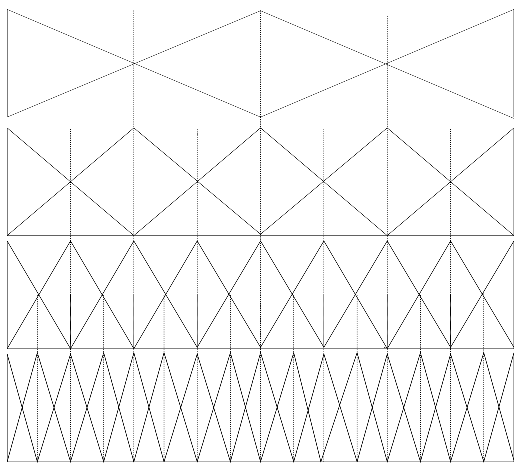

Fig. 2. LH with four levels of granularity 3, 5, 9 and 17, respectively

Generally, the granularity of the linguistic term set at level $t+1$ is obtained from the granularity of the linguistic term set at level t as follows:

$$l(t,n(t)) \rightarrow l(t+1, 2\cdot n(t)-1).$$

2) Computational Model: Herrera et al. in [6] defined the following transformation function $TF^t_{t'}$ between labels of two different levels

$$TF^t_{t'}: l(t,n(t)) \longrightarrow l(t',n(t'))$$

$$TF^t_{t'}(s_i^{n(t)},\alpha^{n(t)}) = \Delta\left(\frac{\Delta^{-1}(s_i^{n(t)},\alpha^{n(t)})\cdot(n(t')-1)}{n(t)-1}\right). \quad (1)$$

$TF^t_{t'}$ is a one-to-one function between the 2-tuple linguistic representation of the different levels of the LH. [6]. The aim of such transformation function is that linguistic terms, independently of its shape and semantic, can be mapped to an unique expression domain, and consequently are amenable to be manipulated with the 2-tuples computational model proposed in [17]. Indeed, in this methodology the membership functions used to modelled a linguistic labels are disregarded because linguistic labels are modelled via their corresponding ordinal 2-tuple representations. A drawback of this methodology regards the high granularity of the last levels of the LH, although this problem has been partially resolved applying the least common multiple approach reported in [18].

III. Type-1 OWA operator to aggregate unbalanced fuzzy linguistic information

Unlike Yager's OWA operator that aggregates crisp values [11], the Type-1 OWA operator is able to aggregate type-1 fuzzy sets with uncertain weights which are also modelled as type-1 fuzzy sets. In the following we present the main aspects of the definition of the type-1 OWA operator (III-A) and its application in aggregating unbalanced fuzzy linguistic information (III-B).

A. The Type-1 OWA operator

As a generalisation of Yager's OWA operator, and based on the extension principle, a Type-1 OWA operator is defined as follows [10]:

Definition 3: Given n linguistic weights $\left\{W^i\right\}_{i=1}^n$ in the form of type-1 fuzzy sets defined on the domain of discourse $[0,1]$, a Type-1 OWA operator is a mapping, Φ,

$$\begin{array}{rcl} \Phi\colon \tilde{P}(\mathbb{R})\times\cdots\tilde{P}(\mathbb{R}) & \longrightarrow & \tilde{P}(\mathbb{R}) \\ (A^1,\cdots,A^n) & \mapsto & Y \end{array}$$

such that

$$\mu_Y(y) = \sup_{\substack{\sum_{k=1}^{n}\bar{w}_i a_{\sigma(i)} = y \\ w_i\in U, a_i\in X}} \begin{pmatrix} \mu_{W^1}(w_1)\wedge\cdots\wedge\mu_{W^n}(w_n) \\ \wedge\mu_{A^1}(a_1)\wedge\cdots\wedge\mu_{A^n}(a_n)\end{pmatrix} \quad (2)$$

where $\bar{w}_i = \dfrac{w_i}{\sum_{i=1}^n w_i}$; σ is a permutation function such that $a_{\sigma(i)} \geq a_{\sigma(i+1)}$, $\forall i = 1,\cdots,n-1$; and $\tilde{P}(\mathbb{R})$ is the set of fuzzy sets on $\mathbb{R}$.

A Direct Approach to performing Type-1 OWA operation was suggested in [10]. Nevertheless, this approach is computationally expensive, which inevitably curtails its further applications to real world decision making. To solve this drawback, a fast approach to Type-1 OWA operations has been developed based on the α-level of fuzzy sets [12].

Definition 4: Given the n linguistic weights $\left\{W^i\right\}_{i=1}^n$ in the form of type-1 fuzzy sets defined on the domain of discourse $[0,1]$, then for each $\alpha\in[0,1]$, an α-level Type-1 OWA operator with α-level weight sets $\left\{W^i_\alpha\right\}_{i=1}^n$ to aggregate the α-level of type-1 fuzzy sets $\left\{A^i\right\}_{i=1}^n$ is given as

$$\Phi_\alpha\left(A^1_\alpha,\cdots,A^n_\alpha\right) = \left\{ \frac{\sum\limits_{i=1}^n w_i a_{\sigma(i)}}{\sum\limits_{i=1}^n w_i} \middle| w_i\in W^i_\alpha,\ a_i\in A^i_\alpha, \forall i\right\} \quad (3)$$

where $W^i_\alpha = \{w|\mu_{W_i}(w)\geq\alpha\}$, $A^i_\alpha = \{x|\mu_{A_i}(x)\geq\alpha\}$, and σ is a permutation function such that $a_{\sigma(i)}\geq a_{\sigma(i+1)}$, $\forall\, i = 1,\cdots,n-1$.

According to the Representation Theorem of type-1 fuzzy sets, the α-level sets $\Phi_\alpha\left(A^1_\alpha,\cdots,A^n_\alpha\right)$ obtained via Definition 4 can be used to construct the following type-1 fuzzy set on $\mathbb{R}$

$$G = \bigcup_{0<\alpha\leq 1} \alpha\Phi_\alpha\left(A^1_\alpha,\cdots,A^n_\alpha\right) \quad (4)$$

with membership function

$$\mu_G(x) = \bigvee_{\alpha : x \in \Phi_\alpha\left(A^1_\alpha, \cdots, A^n_\alpha\right)_\alpha} \alpha \quad (5)$$

Fuzzy sets obtained in (2) and (4) may seem different, however in [12] Zhou et al. proved that both results are equivalent, in what it is known as the *Representation Theorem of Type-1 OWA Operators.*

Theorem 1: Given the n linguistic weights $\left\{W^i\right\}_{i=1}^n$ in the form of type-1 fuzzy sets defined on the domain of discourse $[0,1]$, and the type-1 fuzzy sets $A^1, \cdots, A^n$, then we have that

$$Y = G$$

where Y is the aggregation result defined in (2) and G is the result defined in (4).

Therefore, an effective and practical way of carrying out Type-1 OWA operations is to decompose the Type-1 OWA aggregation into the α-level Type-1 OWA operations and then reconstruct it via the above representation theorem. This α-level approach has been proved to be much faster than the direct approach [12], so it can be used in real time decision making and data mining applications.

When the linguistic weights and the aggregated sets are fuzzy number, the α-level Type-1 OWA operator produces closed intervals [12]:

Theorem 2: Let $\left\{W^i\right\}_{i=1}^n$ be fuzzy numbers on $[0,1]$ and $\left\{A^i\right\}_{i=1}^n$ be fuzzy numbers on $\mathbb{R}$. Then for each $\alpha \in U$, $\Phi_\alpha\left(A^1_\alpha, \cdots, A^n_\alpha\right)$ is a closed interval.

Based on this result, the computation of the Type-1 OWA output according to (4), G, reduces to compute the left end-points and right end-points of the intervals $\Phi_\alpha\left(A^1_\alpha, \cdots, A^n_\alpha\right)$:

$$\Phi_\alpha\left(A^1_\alpha, \cdots, A^n_\alpha\right)_- \text{ and } \Phi_\alpha\left(A^1_\alpha, \cdots, A^n_\alpha\right)_+,$$

where $A^i_\alpha = [A^i_{\alpha-}, A^i_{\alpha+}], W^i_\alpha = [W^i_{\alpha-}, W^i_{\alpha+}]$.

For the left end-points, we have

$$\Phi_\alpha\left(A^1_\alpha, \cdots, A^n_\alpha\right)_- = \min_{\substack{W^i_{\alpha-} \le w_i \le W^i_{\alpha+} \\ A^i_{\alpha-} \le a_i \le A^i_{\alpha+}}} \sum_{i=1}^n w_i a_{\sigma(i)} \Big/ \sum_{i=1}^n w_i \quad (6)$$

while for the right end-points, we have

$$\Phi_\alpha\left(A^1_\alpha, \cdots, A^n_\alpha\right)_+ = \max_{\substack{W^i_{\alpha-} \le w_i \le W^i_{\alpha+} \\ A^i_{\alpha-} \le a_i \le A^i_{\alpha+}}} \sum_{i=1}^n w_i a_{\sigma(i)} \Big/ \sum_{i=1}^n w_i \quad (7)$$

It can be seen that (6) and (7) are programming problems. Solutions to these problems, so that the Type-1 OWA aggregation operation can be performed efficiently, are available from [12].

B. Application of the Type-1 OWA to unbalanced fuzzy linguistic information

This section reports on the results obtained when the Type-1 OWA operator is used to aggregate the unbalanced linguistic information provided in [4] to which the LH methodology approach was already applied. The linguistic information refers to academic assessments of two students on six tests as given in Table I:

TABLE I
STUDENTS' MARKS IN SIX TESTS

	T1	T2	T3	T4	T5	T6
Student1	A	D	D	C	B	A
Student2	D	C	B	C	C	C

In this example, the grading system to evaluate the different tests is the unbalanced linguistic term set $S = \{F, D, C, B, A\}$ with semantic represented via asymmetric triangular fuzzy sets as illustrated in Fig. 1. It is also assumed that the six tests contribute equally in the overall score and therefore same weighting values are used in their aggregation.

As mentioned before, Type-1 OWA operator is able to directly operate with fuzzy numbers even when some of them are modelled with different types of membership functions (triangular, trapezoidal, etc.), and to utilise whole membership functions to carry out the aggregation. Consequently, it is no longer necessary to accomplish any transformation step to operate with linguistic terms that belong to unbalanced linguistic term sets. The aggregated final score for both students are shown in Fig. 3 and Fig. 4, respectively.

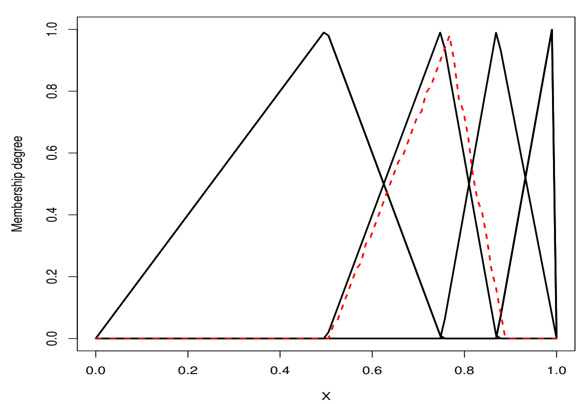

Fig. 3. Student1's marks (solid black lines) and aggregated overall score (dashed red line)

A widely used approach to rank fuzzy numbers consists in converting them into a representative crisp value, and perform the comparison on them, a methodology originally proposed by Zadeh in [19]. This approach has been proposed and used in the selection process of decision making problems under uncertainty where ranking of fuzzy sets is a must to arrive at a decision [20]. Recently, a study by Brunelli and Mezei [21] that compares different ranking methods for fuzzy numbers concludes that 'it is impossible to give a final answer to the question on what ranking method is the best. Most of the time choosing a method rather than another is a matter of preference or is context dependent.' Two defuzzification methods widely used in fuzzy set theory are: the centre of area method (COA) and the mean of maximum method (MOM). The first one computes the centre of mass of the membership function of

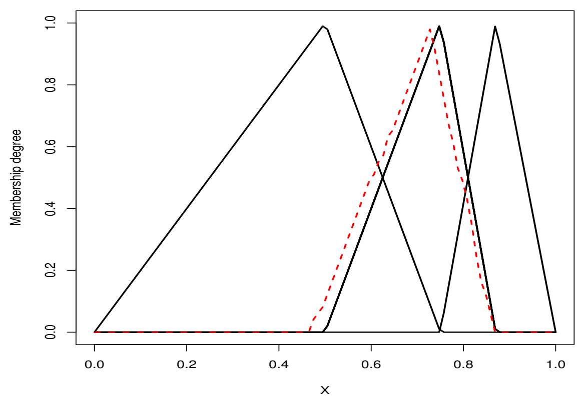

Fig. 4. Student2's marks (solid black lines) and aggregated overall score (dashed red line)

the fuzzy set (the centroid), whereas the second one computes the mid-point of the 1–level set of the fuzzy set. The COA method maintains the underlying semantic ranking relation within the set of linguistic labels, i.e. given two linguistic labels $s_i, s_j \in S$ such that $s_i < s_j$ then $u_{COA}(s_i) < u_{COA}(s_j)$. It is worth mentioning Brunelli and Mezei's correlation study, and their centrality analysis associated to the corresponding correlation network representation, which shows the centre of area method (8) as one of the highest central defuzzification methods.

$$u_{COA}(A) = \frac{\int_x x \cdot \mu_A(x)dx}{\int_x \mu_A(x_i)}. \quad (8)$$

By means of a normalisation process, and following the study presented in [15] it can also be proved that the set of centroids obtained using the Type-1 OWA approach can be mapped with an increasing and continuous function to the set of LH results of symbolic aggregation of the indexes of the set of labels assessed in the unbalanced linguistic term set $S = \{s_0, \ldots, s_g\}$, and therefore they are equivalent. This is corroborated by the results obtained here:

- The Type-1 OWA students' overall scores are also represented as fuzzy numbers with centroid values of 0.72 and 0.68, respectively.
- In both cases, it is obvious that the fuzzy linguistic label $C \in S$ is the closest to the students fuzzy overall scores.
- Herrera et al. [4] overall scores obtained using their LH methodology were the following 2-tuple linguistic values (C,-0.08) and (C,0.16), respectively. Again, the linguistic label $C \in S$ is the closest to the students overall 2-tuple scores, and in the same way as it happens with the Type-1 OWA approach.

IV. Conclusion

In this contribution we present the Type-1 OWA operator as an useful operator to aggregate unbalanced linguistic information. This operator directly works on fuzzy numbers even when some of them are modelled with different types of membership functions (triangular, trapezoidal, etc.), and it also utilises the whole membership functions to compute the aggregated value. The operator has been applied to the same example used by Herrera et al. [4] to show the performance of the LH methodology and it has returned equivalent results. Therefore we can conclude that the Type-1 OWA approach here put forward provides a valid alternative to deal with unbalanced linguistic contexts, and it can be seen as an extension of the linguistic symbolic computational model based on ordinal scales and indexes.

Acknowledgement

This research work has been supported by the research projects TIN2013-40658-P, TIC-05299, TIC-5991, and the University of Granada 'Strengthening through Short-Visits' (Ref. GENIL-SSV 2014) programme.

References

[1] L. Zadeh, "The concept of a linguistic variable and its applications to approximate reasoning," *Information Sciences, Part I, II, III*, vol. 8,8,9, pp. 199–249,301–357,43–80, 1975.
[2] P. P. Bonissone and K. S. Decker, "Selecting uncertainty calculi and granularity: An experiment in trading-off precision and complexity," *Mach Intell Pattern Recognit*, vol. 4, pp. 217–247, 1986.
[3] E. Herrera-Viedma and A. López-Herrera, "A model of information retrieval system with unbalanced fuzzy linguistic information," *International Journal of Intelligent Systems*, vol. 22, no. 11, pp. 1197–2014, 2007.
[4] F. Herrera, E. Herrera-Viedma, and L. Martínez, "A fuzzy linguistic methodology to deal with unbalanced linguistic term sets," *IEEE Transactions on Fuzzy Systems*, vol. 16, no. 2, pp. 354–370, 2008.
[5] O. Cordón, F. Herrera, and I. Zwir, "Linguistic modeling by hierarchical systems of linguistic rules," *IEEE Transactions on Fuzzy Systems*, vol. 10, no. 1, pp. 2–20, 2001.
[6] F. Herrera and L. Martínez, "A model based on linguistic 2-tuples for dealing with multigranularity hierarchical linguistic contexts in multiexpert decision-making," *IEEE Transactions on Systems, Man and Cybernetics. Part B: Cybernetics*, vol. 31, no. 2, pp. 227–234, 2001.
[7] Y. Dong, Y. Xu and S. Yu, "Computing the Numerical Scale of the Linguistic Term Set for the 2-Tuple Fuzzy Linguistic Representation Model", *IEEE Transactions on Fuzzy Systems*, vol. 17, pp. 1366–1378, 2009.
[8] Y. Dong, C. Li and F. Herrera, "Connecting the numerical scale model to the unbalanced linguistic tem sets", *IEEE Int. Conf. Fuzzy Syst.*, pp. 455–462, 2014.
[9] M. Espinilla, J. Liu, and L. Martínez, "An extended hierarchical linguistic model for decision-making problems," *Computational Intelligence*, vol. 27, no. 3, pp. 489–512, 2011.
[10] S. Zhou, F. Chiclana, R. I. John, and J. M. Garibaldi, "Type-1 OWA operators for aggregating uncertain information with uncertain weights induced by type-2 linguistic quantifiers," *Fuzzy Sets and Systems*, vol. 159, no. 24, pp. 3281–3296, 2008.
[11] R. Yager, "On ordered weighted averaging aggregation operators in multicriteria decision making," *IEEE Transactions on Systems, Man and Cybernetics*, vol. 18, pp. 183–190, 1988.
[12] S. Zhou, F. Chiclana, R. I. John, and J. M. Garibaldi, "Alpha-level aggregation: A practical approach to type-1 OWA operation for aggregating uncertain information with applications to breast cancer treatments," *IEEE Transactions on Knowledge and Data Engineering*, vol. 23, no. 10, pp. 1455–1468, 2011.
[13] F. Chiclana and S.-M. Zhou, "Type-reduction of general type-2 fuzzy sets: The type-1 owa approach," *International Journal of Intelligent Systems*, vol. 28, pp. 505–522, 2013.
[14] F. Mata, L. Pérez, S.-M. Zhou, and F. Chiclana, "Type-1 OWA methodology to consensus reaching processes in multi-granular linguistic contexts," *Knowledge-Based Systems*, vol. 58, pp. 11–22, 2014.

[15] P. Pérez-Asurmendi, F. Chiclana, "Linguistic majorities with difference in support," *Applied Soft Computing*, vol. 18, pp. 196 – 208, 2014.

[16] F. Herrera and L. Martínez, "A 2-tuple fuzzy linguistic representation model for computing with words," *IEEE Transactions on Fuzzy Systems*, vol. 8, no. 6, pp. 746–752, 2000.

[17] F. Herrera and L. Martínez, "The 2-tuple linguistic computational model. advantages of its linguistic description, accuracy and consistency," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol. 9, no. Suppl., pp. 33–49, 2001.

[18] M. Espinilla, R. Rodrguez, L. Martnez, F. Mata, and J. Liu, "Optimizing the method for building an extended linguistic hierarchy," 2009, pp. 815–820.

[19] L.A. Zadeh. "Outline of a new approach to the analysis of complex systems and decision processes", *IEEE Transactions on Systems, Man and Cybernetics*, 3, 28–44, 1973.

[20] R. Yager. "OWA aggregation over a continuous interval argument with applications to decision making", *IEEE Transactions on Systems, Man and Cybernetics, Part B: Cybernetics*, 34 (4), 1952–1963, 2004.

[21] M. Brunelli, J. Mezei. "How different are ranking methods for fuzzy numbers? A numerical study", *International Journal of Approximate Reasoning*, 54 (5), 627–639, 2013.